REDUCED-PARAMETER MODELS FOR ANALYSIS OF CAPTURE-RECAPTURE DATA FROM ONE- AND TWO-AGE CLASS OPEN POPULATIONS

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Introduction

Management of any animal population requires some knowledge of population size and of survival and reproductive rates, the two parameters that determine changes in population size. Estimation of population size, survival rate and reproductive rate is thus an extremely important component of wildlife management and research programs.

Efficient methods for estimating survival rate from banding and band recovery data were developed by statisticians and published as early as 1970-71. However, these methods saw very little use until the development of computer programs (ESTIMATE and BROWNIE) to compute these estimates and the publication of the associated monograph by Brownie et al. (1978). Now, largely because of this computer software, these methods are widely used by biologists.

Population size, survival rate and reproductive rate can all be estimated using data from capture-recapture and capture-resighting experiments. A general model for estimation using data from a single age class was published in 1965 (Jolly 1965, Seber 1965). Computer programs are available to provide estimates under this model and the model has thus seen some use by biologists. Age-specific models (Pollock 1981) and reduced-parameter single-age models (Jolly 1982) have been developed but have seen virtually no use because of the absence of readily available computer software.

The following material provides the information needed to develop computer routines for implementing reduced-parameter models which assume either constant survival or constant survival and capture rates. This is presented in two parts. Part 1 relates to the models appropriate when data is available for one age-class only, and Part 2 describes the models for data from young and adults.

The statistical theory underlying the development of these methods is outlined in Jolly (1982) and in Brownie, Hines and Nichols (in preparation).

NOTATION

The notation used is close to that of Jolly (1982) and Seber's text. The correspondence between this notation and symbols in the first page of output from program JOLLY follows.

Jolly (1982)	Program JOLLY	Meaning
m_i	NM	# of marked animals in sample i.
u_i	NU	# of unmarked animals in sample i.
$n_i=m_i+u_i$	NN=NM+NU	Total # caught in sample i.
R_i	NS	# released from sample i.
r_i	R	# caught in sample i and later recaptured.
z_i	Z	# caught before and after but not in sample i.
p_i	P	capture probability.

Some additional notation needed is

s , the number of sampling occasions

d_i , the number of marked animals captured at i but not released ($d_i < m_i$)

$R_{i.}$, the number of animals first caught at i and released at i ($R_{i.} \leq u_i$ and $m_i - d_i = R_i - R_{i.}$)

t_i , the number of time units between samples i and i+1

ϕ , The constant survival rate per unit of time, so that

$\phi^{t_i} = \phi_i$ = survival rate between samples i and i+1,

$q_i = 1-p_i$

$\chi_s = 1$

$1-\chi_i = \phi_i\{1-q_{i+1}\,\chi_{i+1}\}$ for i=1,2,...,s-1.

Here $1-\chi_i$ = probability an animal alive just after sample i is subsequently recaptured.

MODEL B ALGORITHM

Necessary input

m_i , u_i , n_i , R_i , r_i , z_i , t_i , d_i , $R_{i.}$, s

plus "starting values" or initial estimates ϕ^0 and p_i^0 , i=2,...,s.

We should consider two methods for obtaining these starting values.

OPTION 1: First compute (in the main program) the Jolly-Seber estimates

$\hat{\phi}_i$, i=1,...,s-2 , $\hat{p}_i$, i=2,...,s-1 .

Then the starting values are

$$\phi^0 = \frac{\sum_{i=1}^{s-2} t_i \hat{\phi}_i^{1/t_i}}{\sum_{i=1}^{s-2} t_i}$$

$$p_i^0 = \hat{p}_i \quad , \qquad i=2,\ldots,s-1$$

$$p_s^0 = \frac{\sum_{i=2}^{s-1} \hat{p}_i}{s-2}$$

Pass these starting values to the Model B algorithm.

This will usually be the best procedure - except in the case of sparse data sets where problems are encountered in computing the Jolly-Seber estimates $\hat{\phi}_i, \hat{p}_i$. For such data sets, it may be possible to get Model B (or D) estimates. In fact, this is one of the reasons for developing the Model B algorithm. The following option would therefore be useful.

OPTION 2: Before computing the Jolly-Seber estimates (or tests), call the Model B algorithm with starting values ϕ^0 , $p_1^0,\ldots,p_s^0$ supplied by the user in some way (e.g., read in with the data?).

Printed output

Model B estimates $\hat{\phi},\hat{p}_i$, ($\hat{\phi}^{t_i}$ if desired), standard errors, and correlations or covariances.

Also, $\hat{N}_i,\hat{B}_i$ ($\hat{M}_i$?), standard errors, and correlations or covariances.

Goodness of fit test to Model B

Output which need not be printed out

$$1-\hat{\chi}_i \quad , \quad i=1,\ldots,s-1$$

$$\frac{\hat{p}_i}{1-\hat{q}_i\hat{\chi}_i} \quad , \quad i=2,\ldots,s-1$$

These are needed for the test of Model B versus the more general Jolly-Seber Model, and the test of Model D versus Model B.

Computations

First, the iterative procedure is carried out to produce Maximum likelihood (ML) estimates of $\hat{\phi}$, $\hat{\chi}_1,\ldots,\hat{\chi}_{s-1}$. From these estimates (and with $\chi_{s=1}$), we obtain

$$\hat{p}_i = \frac{1}{\hat{\chi}_i}\left[\frac{1-\hat{\chi}_{i-1}}{\hat{\phi}^{t_{i-1}}} - (1-\hat{\chi}_i)\right] \quad , \qquad i=2,\ldots,s \ .$$

Lastly, $\hat{M}_i$, $\hat{N}_i$, $\hat{B}_i$ are obtained.

Some notation

Recall $\chi_s = 1$ (by definition).

$$1-\chi_i = \phi^{t_i}(1-q_{i+1}\ \chi_{i+1}) \quad , \qquad i=1,\ldots,s-1,$$

computed recursively.

Given starting values ϕ^0 , $p_2^0,\ldots,p_s^0$, calculate

$$q_i^0 = 1-p_i^0 \ , \qquad i=2,\ldots,s \ , \qquad \text{and}$$

$$1-\chi_i^0 = (\phi^0)^{t_i}(1-q_{i+1}^0\ \chi_{i+1}^0) \quad , \quad \text{computed recursively,}$$

working backwards from $\chi_s^0 = 1$, $1-\chi_{s-1}^0 = (\phi^0)^{t_{s-1}}(1-q_s^0\ \chi_s^0)$, etc., to $1-\chi_1^0$.

Then, $\underset{\sim}{\theta}^0 = \begin{pmatrix} \phi^0 \\ \chi_1^0 \\ \cdot \\ \cdot \\ \cdot \\ \chi_{s-1}^0 \end{pmatrix}$ is an $s\times1$ vector of starting values for the iteration procedure.

$\underset{\sim}{\theta}^n$ represents the corresponding vector at the n<u>th</u> iteration, n=1,2,3,...

θ_i^n is the i<u>th</u> element of $\underset{\sim}{\theta}^n$

$\underset{\sim}{\Delta}^n$ is an s×1 vector of first order partial derivatives evaluated using elements in $\underset{\sim}{\theta}^n$, n=0,1,2,...

Δ_i^n is the i<u>th</u> element of $\underset{\sim}{\Delta}^n$

H^n is an s×s matrix, involving second order partials, evaluated using elements in $\underset{\sim}{\theta}^n$, n=0,1,2,...

$V^n = (H^n)^{-1}$ is related to the variance-covariance matrix

v_{ij}^n is the element in the i<u>th</u> row, j<u>th</u> column of V^n .

Explicit formulas for computing Δ^0 , H^0 , Δ^1 , H^1 , etc., are given below, but first the basic iteration scheme is outlined.

<u>ITERATION SCHEME</u>

Compute $\underset{\sim}{\Delta}^0$, H^0 , using the starting values in $\underset{\sim}{\theta}^0$ (i.e., using ϕ^0 , $\chi_1^0,\ldots,$ χ_{s-1}^0 , and $p_2^0,\ldots,p_s^0$).

Compute $V^0 = (H^0)^{-1}$.

(NOTE that this involves inverting a symmetric matrix and requires an efficient and numerically accurate algorithm.)

1st iteration

Compute $\underset{\sim}{\theta}^1 = \begin{pmatrix} \phi^1 \\ \chi_1^1 \\ \cdot \\ \cdot \\ \cdot \\ \chi_{s-1}^1 \end{pmatrix}$.

In matrix notation,

$$\underset{\sim}{\theta}^1 = \underset{\sim}{\theta}^0 + V^0 \underset{\sim}{\Delta}^0 \quad .$$

Specifically,

$$\theta_1^1 = \phi_1^1 = \phi_1^0 + \sum_{j=1}^{s} V_{ij}^0 \Delta_j^0$$

$$\theta_i^1 = \chi_{i-1}^1 = \chi_{i-1}^0 + \sum_{j=1}^{s} V_{ij}^0 \Delta_j^0 \quad , \qquad i=2,\ldots,s,$$

Using $\underset{\sim}{\theta}^1$ and $\underset{\sim}{\theta}^0$, check for convergence (see below). If convergence criterion is not satisfied, compute $p_2^1,\ldots,p_s^1$, using elements in $\underset{\sim}{\theta}^1$, by

$$p_i^1 = \frac{1}{\chi_i^1} \left[\frac{1-\chi_{i-1}^1}{(\phi^1)^{t_{i-1}}} - (1-\chi_i^1) \right] \quad .$$

Use these to compute $\underset{\sim}{\Delta}^1$, and H^1 .

Compute $V^1 = (H^1)^{-1}$.

2nd iteration

Compute $\underset{\sim}{\theta}^2 = \underset{\sim}{\theta}^1 + V^1\Delta^1$, and check for convergence using $\underset{\sim}{\theta}^2$ and $\underset{\sim}{\theta}^1$.

nth iteration

Continue iterating in this way until convergence criterion is satisfied. That is, at the nth iteration, n=1,2,...

$$\underset{\sim}{\theta}^n = \underset{\sim}{\theta}^{n-1} + V^{n-1}\ \underset{\sim}{\Delta}^{n-1} \quad ,$$

and convergence is declared if

$$|\theta_i^n - \theta_i^{n-1}| < 10^{-5} \quad \text{for all } i,\ i=1,2,\ldots,s.$$

If convergence occurs at the nth iteration, $\underset{\sim}{\theta}^{n-1}$ is the vector of ML estimates and V^{n-1} the estimated variance-covariance matrix for the parameters ϕ , $\chi_1,\ldots,\chi_{s-1}$. MLe's and variances for $p_2,\ldots,p_s$ are obtained from $\underset{\sim}{\theta}^{n-1}$, V^{n-1} (see below).

If the convergence criterion is not met at the nth iteration (i.e., $|\theta_i^n - \theta_i^{n-1}| \geq 10^{-5}$ for any i), then compute $\underset{\sim}{\Delta}^n$, V^n and go to the (n+1)st iteration.

NOTE: The convergence criterion 10^{-5} is arbitrary. I could not find a program listing for ESTIMATE or BROWNIE to see what was used there. The number of iterations should be limited to 20 or 25, and nonconvergence declared if this would be exceeded.

Formulas for computing $\underset{\sim}{\Delta}^n$

If L represents the log likelihood function under Model B, the elements of $\underset{\sim}{\Delta}$ are the first order partials $\delta L/\delta\theta_i$. Thus, $\Delta_1 = \delta L/\delta\phi$, $\Delta_i = \delta L/\delta\chi_{i-1}$, $i=2,\ldots,s$, given below.

$$\Delta_1 = \frac{1}{\phi}\sum_{i=2}^{s-1}\frac{t_{i-1}}{\chi_i}\left\{\frac{z_i}{q_i} - \frac{m_i(1-\chi_i)}{p_i}\right\}$$

$$\Delta_2 = \frac{R_1-r_1}{\chi_1} - \frac{1}{\phi^{t_1}q_2\chi_2}\left(\frac{m_2}{p_2} - m_2 - z_2\right) .$$

$$\begin{pmatrix}\text{OMIT if}\\ s=3\end{pmatrix}\qquad \Delta_i = \frac{1}{\chi_{i-1}}\left(\frac{q_{i-1}\,m_{i-1}}{p_{i-1}} - z_{i-1}+R_{i-1}-r_{i-1}\right) - \frac{1}{\phi^{t_{i-1}}\,q_i\chi_i}\left(\frac{q_i m_i}{p_i} - z_i\right)$$

$$i=3,\ldots,s-1$$

$$\Delta_s = \frac{1}{\chi_{s-1}}\left(\frac{q_{s-1}\,m_{s-1}}{p_{s-1}} - z_{s-1}+R_{s-1}-r_{s-1}\right) - \frac{m_s}{1-\chi_{s-1}}$$

To compute the elements of $\underset{\sim}{\Delta}^0$, in the above expressions for Δ_i , replace ϕ,p_i,q_i,χ_i with $\phi^0,p_i^0,q_i^0,\chi_i^0$, respectively. Similarly, $\underset{\sim}{\Delta}^1$ is computed using $\phi^1,p_i^1,q_i^1,\chi_i^1$ (produced after the 1st iteration). And so on for $\underset{\sim}{\Delta}^n$, $n=2,3,\ldots$.

Formulas for computing H^n

The elements of H correspond to $-\delta^2 L/\delta\theta_i\delta\theta_j$, so H is of dimension s×s and is symmetric, i.e., $H_{ij} = H_{ji}$. Formulas for elements in the diagonal and upper triangle are given below, the rest are obtained by symmetry.

Elelents in row 1 are:

$$H_{11} = \frac{1}{\phi^2} \sum_{i=2}^{s-1} \frac{t_{i-1}^2}{\chi_i^2} \left(1-q_i\chi_i\right) \left\{\frac{z_i}{q_i^2} + \frac{m_i}{p_i^2} \left(1-\chi_i\right)\right\}$$

$$H_{12} = \frac{1}{\phi} \frac{t_1}{\chi_2^2 \, \phi^{t_1}} \left\{\frac{z_2}{q_2^2} + \frac{m_2(1-\chi_2)}{p_2^2}\right\}$$

$\left(\begin{array}{l}\text{OMIT if} \\ s=3\end{array}\right)$

$$H_{1j} = - \frac{t_{j-2} m_{j-1} (1-q_{j-1}\chi_{j-1})}{\phi \, p_{j-1}^2 \, \chi_{j-1}^2} + \frac{t_{j-1}}{\phi \, \chi_j^2 \, \phi^{t_{j-1}}} \left\{\frac{z_j}{q_j^2} + \frac{m_j(1-\chi_j)}{p_j^2}\right\}$$

$j=3,\ldots s-1$.

$$H_{1s} = - \frac{t_{s-2} m_{s-1} (1-q_{s-1}\chi_{s-1})}{\phi \, p_{s-1}^2 \, \chi_{s-1}^2} \quad .$$

Rows 2 to s:

$$H_{22} = \frac{R_1 - r_1}{\chi_1^2} + \frac{1}{(\phi^{t_1}\ \chi_2)^2} \left\{ \frac{m_2}{p_2^2} + \frac{z_2}{q_2^2} \right\}$$

$\left(\begin{matrix}\text{OMIT if} \\ s=3\end{matrix}\right)$

$$H_{ii} = \frac{1}{\chi_{i-1}^2} \left\{ \frac{m_{i-1}}{p_{i-1}^2} - m_{i-1} - z_{i-1} + R_{i-1} - r_{i-1} \right\} + \frac{1}{(\phi^{t_{i-1}}\ \chi_i)^2} \left(\frac{m_i}{p_i^2} + \frac{z_i}{q_i^2} \right)$$

$i=3,\ldots,s-1$

$$H_{ss} = \frac{1}{\chi_{s-1}^2} \left(\frac{m_{s-1}}{p_{s-1}^2} - m_{s-1} - z_{s-1} + R_{s-1} - r_{s-1} \right) + \frac{m_s}{(1-\chi_{s-1})^2}$$

$$H_{i,i+1} = - \frac{m_i}{\phi^{t_{i-1}} (p_i \chi_i)^2} \qquad i=2,3,\ldots,s-1$$

$$H_{i,j} = 0 \qquad i=2,\ldots,s-2, \quad j>i+1 \ .$$

Elements of H^n, $n=0,1,2,\ldots$ are evaluated by substituting $\phi^n, \chi_i^n, p_i^n, q_i^n$ for the parameters ϕ, χ_i, p_i, q_i in the above expressions for H_{ij} .

$V^n = (H^n)^{-1}$ is computed by inverting the symmetric matrix H^n .

Printing Model B estimates $\hat{\phi}$, $\hat{p}_2, \ldots, \hat{p}_s$.

[Question: Do we need to print $(\hat{\phi})^{t_1}, \ldots, (\hat{\phi})^{t_{s-1}}$ in addition to $\hat{\phi}$?]

After convergence, $\underset{\sim}{\theta}$ contains the MLe's $\hat{\phi}$, $\hat{\chi}_1, \ldots, \hat{\chi}_{s-1}$ and V the estimated variances-covariances.

$$\hat{\phi} = \theta_1 \quad , \qquad \hat{\chi}_i = \theta_{i+1} \quad , \quad i=1,\ldots,s-1 .$$

$$V_{11} = \mathrm{Var}(\hat{\phi}) \quad , \qquad V_{i+1,i+1} = \mathrm{Var}(\hat{\chi}_i) \quad , \quad i=1,\ldots,s-1 .$$

$$V_{1j} = \mathrm{Cov}(\hat{\phi},\hat{\chi}_{j-1}) \quad , \qquad V_{i,i+1} = \mathrm{Cov}(\hat{\chi}_{i-1},\hat{\chi}_i) \quad .$$

From $\underset{\sim}{\theta}$, V , obtain

$$\hat{p}_i = \frac{1}{\hat{\chi}_i}\left[\frac{1-\hat{\chi}_{i-1}}{\hat{\phi}^{t_{i-1}}} - (1-\hat{\chi}_i)\right] \qquad i=2,\ldots,s .$$

(Note for $\hat{p}_s$, define $\hat{\chi}_s = 1$, and the above formula works.)

$$\mathrm{Var}(\hat{p}_i) = \frac{1}{\chi_i^2}\left[q_i^2\,\mathrm{Var}(\hat{\chi}_i) + \frac{\mathrm{Var}(\hat{\chi}_{i-1})}{(\phi^{t_{i-1}})^2} + \frac{t_{i-1}^2(1-q_i\chi_i)^2}{\phi^2}\,\mathrm{Var}(\hat{\phi})\right.$$

$$- \frac{2q_i}{\phi^{t_{i-1}}}\,\mathrm{Cov}(\hat{\chi}_{i-1},\hat{\chi}_i) + \frac{2t_{i-1}(1-q_i\chi_i)}{\phi\,\phi^{t_{i-1}}}\,\mathrm{Cov}(\hat{\phi},\hat{\chi}_{i-1})$$

$$\left.- \frac{2q_i t_{i-1}(1-q_i\chi_i)}{\phi}\,\mathrm{Cov}(\hat{\phi},\hat{\chi}_i)\right] \qquad i=2,\ldots,s-1 .$$

$$\mathrm{Var}(\hat{p}_s) = \frac{\mathrm{Var}(\hat{\chi}_{s-1})}{(\phi^{t_{s-1}})^2} + \frac{t_{s-1}^2\, p_s^2}{\phi^2}\,\mathrm{Var}\,\hat{\phi} + \frac{2t_{s-1}p_s}{\phi\,\phi^{t_{s-1}}}\,\mathrm{Cov}(\hat{\phi},\hat{\chi}_{s-1}) \quad ,$$

where $\mathrm{Var}(\hat{p}_i)$ is evaluated by substituting $\hat{\chi}_i$ for χ_i , $\hat{q}_i$ for q_i , etc., and V_{11} for Var $\hat{\phi}$, etc.

$\mathrm{Var}(\hat{p}_i)$ and covariances which must be computed are given below in terms of the elements of V, and with ϕ^{t_i} written as ϕ_i . (Recall $\hat{\phi}_i = (\hat{\phi})^{t_i}$) .

$$\mathrm{Var}(\hat{p}_i) = \frac{1}{\chi_i^2}\left[q_i^2\, V_{i+1,i+1} + \frac{V_{ii}}{\phi_{i-1}^2} + \frac{t_{i-1}^2(1-q_i\chi_i)^2}{\phi^2}V_{11} - \frac{2q_i}{\phi_{i-1}}V_{i,i+1} + \frac{2t_{i-1}(1-q_i\chi_i)}{\phi\,\phi_{i-1}}V_{1i} - \frac{2q_i t_{i-1}(1-q_i\chi_i)}{\phi}V_{1,i+1}\right]$$

$$i=2,\ldots,s-1 \qquad (1)$$

$$\mathrm{Var}(\hat{p}_s) = \frac{V_{ss}}{(\phi_{s-1})^2} + \frac{t_{s-1}^2 p_s^2}{\phi^2}V_{11} + \frac{2t_{s-1}p_s}{\phi\,\phi_{s-1}}V_{1s} \qquad (2)$$

$$\mathrm{Cov}(\hat{p}_i,\hat{p}_j) = \frac{1}{\chi_i\chi_j}\left[\frac{t_{i-1}(1-q_i\chi_i)}{\phi}\left\{\frac{t_{j-1}(1-q_j\chi_j)}{\phi}V_{11} + \frac{V_{1j}}{\phi_{j-1}} - q_jV_{1,j+1}\right\} + \frac{1}{\phi_{i-1}}\left\{\frac{t_{j-1}(1-q_j\chi_j)}{\phi}V_{1i} + \frac{V_{ij}}{\phi_{j-1}} - q_jV_{i,j+1}\right\} - q_i\left\{\frac{t_{j-1}(1-q_j\chi_j)}{\phi}V_{1,i+1} + \frac{V_{i+1,j}}{\phi_{j-1}} - q_jV_{i+1,j+1}\right\}\right]$$

$$2\le i,j<s \qquad (3)$$

$$\mathrm{Cov}(\hat{p}_i,\hat{p}_s) = \frac{1}{\chi_{s-1}}\left[\frac{t_{i-1}(1-q_i\,\chi_i)}{\phi}\left\{\frac{t_{s-1}p_s}{\phi}V_{11} + \frac{V_{1s}}{\phi_{s-1}}\right\}\right.$$

$$+ \frac{1}{\phi_{i-1}}\left\{\frac{t_{s-1}p_s}{\phi}V_{1,i} + \frac{V_{i,s}}{\phi_{s-1}}\right\}$$

$$\left. - q_i\left\{\frac{t_{s-1}p_s}{\phi}V_{1,i+1} + \frac{V_{i+1,s}}{\phi_{s-1}}\right\}\right] \qquad i=2,\ldots,s-1 \tag{4}$$

NOTE: $\mathrm{Cov}(\hat{p}_i,\hat{p}_i) = \mathrm{Var}(\hat{p}_i)$, so that (3) with j=i, and (1), should yield the same numerical result for a given value of i . Also note that $\mathrm{Cov}(\hat{p}_i,\hat{p}_j) = \mathrm{Cov}(\hat{p}_j,\hat{p}_i)$.

$$\mathrm{Cov}(\hat{\phi},\hat{p}_i) = \frac{1}{\chi_i}\left[\frac{-t_{i-1}(1-q_i\chi_i)}{\phi}V_{11} - \frac{V_{1i}}{\phi_{i-1}} + q_iV_{1,i+1}\right] \tag{5}$$

$i=2,\ldots,s-1$

$$\mathrm{Cov}(\hat{\phi},\hat{p}_s) = \frac{-t_{s-1}p_s}{\phi}V_{11} - \frac{V_{1s}}{\phi_{s-1}} \tag{6}$$

Model B Estimates of M_i , N_i , B_i , Variances and Covariances

$$\hat{M}_i = \frac{m_i+z_i}{1-\hat{q}_i\hat{\chi}_i} \quad , \qquad i=2,\ldots,s \qquad (7a)$$

$$\mathcal{m}_2 = \hat{\phi}_1 R_1 \quad , \qquad \mathcal{m}_{i+1} = \hat{\phi}_i\left\{\mathcal{m}_i - \hat{p}_i \frac{d_i}{m_i}\mathcal{m}_i + R_i.\right\} \qquad i=2,\ldots,s-1 \qquad (7b)$$

$$\hat{U}_i = \frac{u_i}{\hat{p}_i} \qquad i=2,\ldots,s \qquad (8)$$

$$\hat{N}_i = \hat{U}_i + \hat{M}_i \qquad i=2,\ldots,s \qquad (9)$$

$$\hat{B}_i = \hat{U}_{i+1} - \hat{\phi}_i\hat{q}_i\hat{U}_i \qquad i=2,\ldots,s-1 \qquad (10)$$

NOTE: Define $z_s=0$, and, as before, $\hat{\chi}_s \equiv 1$, so that (7a) holds for $i=s$.

To compute variances/covariances for $\hat{M}_i$, $\hat{N}_i$, $\hat{B}_i$, first compute the matrices X and W defined below, replacing the parameters ϕ , ϕ_i , q_i , χ_i , etc. by the estimates $\hat{\phi}$, $\hat{\phi}_i$, $\hat{q}_i$, $\hat{\chi}_i$, etc.

$$\text{Define} \quad \alpha_{kj} = \begin{cases} 1 & \text{if } k=j \\ q_{k+1}\,\phi_{k+1}\cdots q_j\,\phi_j = \prod_{\ell=k+1}^{j} q_\ell\phi_\ell & \text{if } k<j \end{cases}$$

Compute α_{kj} for $j=1,\ldots,s-1$ and $k=1,\ldots,j$.

Row 1 of the s by (s-1) matrix X

$$X_{11} = 0$$

$$X_{1j} = \frac{q_{j+1}\ \chi_{j+1}}{\phi} \sum_{k=2}^{j} \frac{t_{k-1}\ \phi_k}{\chi_k} M_k\ \alpha_{kj} \qquad j=2,\ldots,s-1 \tag{11}$$

Rows 2 to s of X

$$X_{i,i-1} = \frac{-q_i \chi_i M_i}{\chi_{i-1}(1-\chi_{i-1})} \qquad i=2,\ldots,s \tag{12}$$

$$X_{ij} = \begin{cases} 0 & j=1,\ldots,i-2;\ i=3,\ldots,s \\ \dfrac{q_{j+1}\chi_{j+1}(1-\phi_{i-1})M_i\phi_i\alpha_{ij}}{\phi_{i-1}\chi_{i-1}\chi_i} & j=i,i+1,\ldots,s-1;\ i=2,\ldots,s-1 \end{cases} \tag{13}$$

For example, if $s=4$,

$$X = \begin{bmatrix} 0 & \dfrac{q_3\chi_3}{\phi}\dfrac{t_1 M_2 \phi_2}{\chi_2} & \dfrac{q_4\chi_4}{\phi}\displaystyle\sum_{k=2}^{3}\dfrac{t_{k-1}\phi_k M_k \alpha_{k3}}{\chi_k} \\ \dfrac{-q_2\chi_2 M_2}{\chi_1(1-\chi_1)} & \dfrac{q_3\chi_3(1-\phi_1)M_2\phi_2}{\phi_1\chi_1\chi_2} & \dfrac{q_4\chi_4(1-\phi_1)M_2\phi_2\alpha_{23}}{\phi_1\chi_1\chi_2} \\ 0 & \dfrac{-q_3\chi_3 M_3}{\chi_2(1-\chi_2)} & \dfrac{q_4\chi_4(1-\phi_2)M_3\phi_3}{\phi_2\chi_2\chi_3} \\ 0 & 0 & \dfrac{-q_4\chi_4 M_4}{\chi_3(1-\chi_3)} \end{bmatrix}$$

Next, compute $W = VX$ by matrix multiplication, so that W is s by $(s-1)$. Elements of W are denoted W_{ij} in the formulae below.

$$\mathrm{Var}(\hat{M}_i) = M_i^2 \left\{ \frac{q_i \chi_i}{m_i + z_i} + \frac{t_{i-1}^2}{\phi^2} V_{11} + \frac{V_{ii}}{(1-\chi_{i-1})^2} + \frac{2t_{i-1}}{\phi(1-\chi_{i-1})} V_{1i} + \frac{2t_{i-1}}{\phi M_i} W_{1,i-1} + \frac{2}{M_i(1-\chi_{i-1})} W_{i,i-1} \right\} \qquad i=2,\ldots,s \tag{14}$$

$$\begin{aligned}\mathrm{Cov}(\hat{M}_i,\hat{M}_j) = {} & \frac{q_j \chi_j M_i \alpha_{i-1,j-1}}{1-q_i\chi_i} + \frac{M_i t_{i-1}}{\phi} W_{1,j-1} + \frac{M_i}{1-\chi_{i-1}} W_{i,j-1} \\ & + \frac{M_j t_{j-1}}{\phi} \left\{ W_{1,i-1} + \frac{M_i t_{i-1}}{\phi} V_{11} + \frac{M_i}{1-\chi_{i-1}} V_{1i} \right\} \\ & + \frac{M_j}{1-\chi_{j-1}} \left\{ W_{j,i-1} + \frac{M_i t_{i-1}}{\phi} V_{1j} + \frac{M_i}{1-\chi_{i-1}} V_{ji} \right\}\end{aligned}$$

$$j=i+1,\ldots,s; \; i=2,\ldots,s-1 \tag{15}$$

$$\begin{aligned}\mathrm{Cov}(\hat{p}_i,\hat{M}_j) = \frac{1}{\chi_i} \Bigg[& q_i W_{i+1,j-1} - \frac{t_{i-1}(1-q_i\chi_i)}{\phi} W_{1,j-1} - \frac{W_{i,j-1}}{\phi_{i-1}} \\ & + \frac{t_{j-1}M_j}{\phi} \left\{ q_i V_{1,i+1} - \frac{t_{i-1}(1-q_i\chi_i)}{\phi} V_{11} - \frac{V_{1i}}{\phi_{i-1}} \right\} \\ & + \frac{M_j}{1-\chi_{j-1}} \left\{ q_i V_{j,i+1} - \frac{t_{i-1}(1-q_i\chi_i)}{\phi} V_{1j} - \frac{V_{ij}}{\phi_{i-1}} \right\} \Bigg]\end{aligned}$$

$$i=2,\ldots,s-1, \; j=2,\ldots,s \tag{16}$$

$$\mathrm{Cov}(\hat{p}_s,\hat{M}_j) = \frac{-t_{s-1}p_s}{\phi} W_{1,j-1} - \frac{W_{s,j-1}}{\phi_{s-1}} - \frac{t_{j-1}M_j}{\phi}\left\{\frac{t_{s-1}p_s}{\phi} V_{11} + \frac{V_{1s}}{\phi_{s-1}}\right\}$$

$$- \frac{M_j}{1-\chi_{j-1}}\left\{\frac{t_{s-1}p_s}{\phi} V_{1j} + \frac{V_{sj}}{\phi_{s-1}}\right\} \qquad j=2,\ldots,s \qquad (17)$$

NOTE: The variances and covariances in (1) through (10), and (14) through (17) are used in computing $\mathrm{Var}(\hat{N}_i)$, $\mathrm{Var}(\hat{B}_i)$, and covariances. It is probably not necessary to print out $\mathrm{Var}(\hat{M}_i)$, $\mathrm{Cov}(\hat{M}_i,\hat{M}_j)$, $\mathrm{Cov}(\hat{p}_i,\hat{M}_j)$.

$$\mathrm{Var}(\hat{N}_i) = \mathrm{Var}(\hat{M}_i) + \frac{U_i}{p_i}\left\{q_i + \frac{U_i}{p_i}\mathrm{Var}(\hat{p}_i) - 2\mathrm{Cov}(\hat{p}_i,\hat{M}_i)\right\}$$

$$i=2,\ldots,s \qquad (18)$$

$$\mathrm{Cov}(\hat{N}_i,\hat{N}_j) = \mathrm{Cov}(\hat{M}_i,\hat{M}_j) - \frac{U_i}{p_i}\mathrm{Cov}(\hat{p}_i,\hat{M}_j) - \frac{U_j}{p_j}\mathrm{Cov}(\hat{p}_j,\hat{M}_i)$$

$$+ \frac{U_i}{p_i}\frac{U_j}{p_j}\mathrm{Cov}(\hat{p}_i,\hat{p}_j) \qquad i=2,\ldots,s-1;\ j=i+1,\ldots,s \qquad (19)$$

$$\mathrm{Var}(\hat{B}_i) = \frac{U_{i+1}\,q_{i+1}}{p_{i+1}} + q_i\phi_iU_i\left(1 + \frac{q_i\phi_i}{p_i}\right)$$

$$+ \left(\frac{q_i\phi_it_iU_i}{\phi}\right)^2 V_{11} + \left(\frac{\phi_iU_i}{p_i}\right)^2 \mathrm{Var}(\hat{p}_i)$$

$$+ \left(\frac{U_{i+1}}{p_{i+1}}\right)^2 \mathrm{Var}(\hat{p}_{i+1}) - \frac{2q_it_i\phi_i^2U_i^2}{\phi p_i}\mathrm{Cov}(\hat{\phi},\hat{p}_i)$$

$$+ \frac{2q_i\phi_it_iU_iU_{i+1}}{\phi p_{i+1}}\mathrm{Cov}(\hat{\phi},\hat{p}_{i+1}) - \frac{2\phi_iU_iU_{i+1}}{p_ip_{i+1}}\mathrm{Cov}(\hat{p}_i,\hat{p}_{i+1})$$

$$i=2,\ldots,s-1 \qquad (20)$$

(OMIT if s=3)

$$\mathrm{Cov}(\hat{B}_{i-1},\hat{B}_i) = -\frac{q_i\phi_i U_i}{p_i} + \frac{q_{i-1}\phi_{i-1}t_{i-1}U_{i-1}}{\phi}\left\{\frac{q_i\phi_i t_i U_i}{\phi} V_{11} - \frac{\phi_i U_i}{p_i}\mathrm{Cov}(\hat{\phi},\hat{p}_i) + \frac{U_{i+1}}{p_{i+1}}\mathrm{Cov}(\hat{\phi},\hat{p}_{i+1})\right\}$$

$$+ \frac{\phi_{i-1}U_{i-1}}{p_{i-1}}\left\{\frac{-q_i\phi_i t_i U_i}{\phi}\mathrm{Cov}(\hat{\phi},\hat{p}_{i-1}) + \frac{\phi_i U_i}{p_i}\mathrm{Cov}(\hat{p}_{i-1},\hat{p}_i) - \frac{U_{i+1}}{p_{i+1}}\mathrm{Cov}(\hat{p}_{i-1},\hat{p}_{i+1})\right\}$$

$$+ \frac{U_i}{p_i}\left\{\frac{q_i\phi_i t_i U_i}{\phi}\mathrm{Cov}(\hat{\phi},\hat{p}_i) - \frac{\phi_i U_i}{p_i}\mathrm{Var}(\hat{p}_i) + \frac{U_{i+1}}{p_{i+1}}\mathrm{Cov}(\hat{p}_i,\hat{p}_{i+1})\right\} \qquad i=3,\ldots,s-1 \qquad (21)$$

(OMIT if s=3 or 4)

$$\mathrm{Cov}(\hat{B}_i,\hat{B}_j) = \frac{q_i\phi_i t_i U_i}{\phi}\left\{\frac{q_j\phi_j t_j U_j}{\phi} V_{11} - \frac{\phi_j U_j}{p_j}\mathrm{Cov}(\hat{\phi},\hat{p}_j) + \frac{U_{j+1}}{p_{j+1}}\mathrm{Cov}(\hat{\phi},\hat{p}_{j+1})\right\} + \frac{\phi_i U_i}{p_i}\left\{\frac{-q_j\phi_j t_j U_j}{\phi}\mathrm{Cov}(\hat{\phi},\hat{p}_i) + \frac{\phi_j U_j}{p_j}\mathrm{Cov}(\hat{p}_i,\hat{p}_j) - \frac{U_{j+1}}{p_{j+1}}\mathrm{Cov}(\hat{p}_i,\hat{p}_{j+1})\right\}$$

$$+ \frac{U_{i+1}}{p_{i+1}}\left\{\frac{q_j\phi_j t_j U_j}{\phi}\mathrm{Cov}(\hat{\phi},\hat{p}_{i+1}) - \frac{\phi_j U_j}{p_j}\mathrm{Cov}(\hat{p}_{i+1},\hat{p}_j) + \frac{U_{j+1}}{p_{j+1}}\mathrm{Cov}(\hat{p}_{i+1},\hat{p}_{j+1})\right\}$$

$$i=2,\ldots,s-3,\ j=i+2,\ldots,s-1 \qquad (22)$$

MODEL D ALGORITHM

Necessary input

As for Model B, with starting values via OPTION 1 or OPTION 2.

OPTION 1: Starting values are ϕ^0 = Model B estimate $\hat{\phi}$,

$$p^0 = \frac{1}{s-1} \sum_{i=2}^{s} \hat{p}_i \quad ,$$

where $\hat{\phi}$ and $\hat{p}_2,\ldots,\hat{p}_s$ are the Model B estimates.

OPTION 2: Read in values for ϕ^0 , p^0 .

Printed output

Model D estimates $\hat{\phi}$ and $\hat{p}$, ($\hat{\phi}^{t_i} = \hat{\phi}_i$ if desired), standard errors, and covariances. Also, Model D estimates $\hat{N}_i$, $\hat{B}_i$ ($\hat{M}_i$?), standard errors, covariances.

Computations

The iterative procedure produces ML estimates $\hat{\phi}$ and $\hat{p}$, and the (estimated) variance-covariance matrix V (dimension 2 by 2). At each iteration, it is necessary to compute

$$\left.\begin{aligned} &1-\chi_i = \phi_i\{1-q\chi_{i+1}\} \ , && i=2,\ldots,s-1 \quad \text{with } \phi_i = \phi^{t_i} \ , \\ & && q=1-p \ , \text{ and } \chi_s \equiv 1 \text{ as before;} \\ &\gamma_1 \equiv 0 \ , && \\ &\gamma_{i+1} = \phi_i\left\{q\gamma_i + \frac{R_i - r_i}{\chi_i}\right\} && i+1=2,\ldots,s \quad ; \end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned} w_1 &\equiv 0 \\ w_{i+1} &= \phi_i^2\left\{q^2 w_i + \frac{R_i}{\chi_i}\right\} , \qquad i+1 = 2,\ldots,s \end{aligned}\right\} \tag{24}$$

and

α_{kj} , $j=1,\ldots,s-1$, $k=1,\ldots,j$ as on page 14.

Iteration Scheme

Iteration proceeds as for Model B but with the vector of starting values $\underset{\sim}{\theta}^0 = \begin{pmatrix}\phi^0 \\ p^0\end{pmatrix}$, and Δ (2×1) and H (2×2) as defined below.

Formulas for computing $\underset{\sim}{\Delta}$

$$\Delta_1 = \frac{1}{\phi}\sum_{i=2}^{s} t_{i-1}\left\{m_i + z_i - \gamma_i(1-q\chi_i)\right\} \tag{25}$$

$$\Delta_2 = \sum_{i=2}^{s}\left\{\frac{m_i}{p} - \frac{z_i}{q} - \chi_i\gamma_i\right\} . \tag{26}$$

At the nth step, $\underset{\sim}{\Delta}^n$ is computed by substituting the current values ϕ^n , p^n , χ_i^n , γ_i^n into the above formulas.

Formulas for computing H (2×2)

$$H_{11} = \frac{1}{\phi^2} \sum_{i=2}^{s} t_{i-1}^2 \left(1-q\chi_i\right) \left\{\gamma_i + (1-q\chi_i)w_i\right\}$$

$$+ \frac{2}{\phi^2} \sum_{i=3}^{s} t_{i-1} \left(1-q\chi_i\right) \sum_{j=2}^{i-1} t_{j-1}\alpha_{j-1,i-1} \left\{\gamma_j + (1-q\chi_j)w_j\right\} \tag{27}$$

$$H_{12} = H_{21} = \frac{1}{\phi} \sum_{i=2}^{s} t_{i-1} \left\{(1-q\chi_i)w_i + \gamma_i\right\} \sum_{j=i}^{s} \alpha_{i-1,j-1}\, \chi_j$$

$$+ \frac{1}{\phi} \sum_{i=3}^{s} t_{i-1} \left(1-q\chi_i\right) \sum_{j=2}^{i-1} \phi_j \alpha_{j,i-1} \left(q\chi_j w_j - \gamma_j\right) \tag{28}$$

$$H_{22} = \sum_{i=2}^{s} \left\{\frac{m_i}{p^2} + \frac{z_i}{q^2} + \chi_i^2 w_i\right\}$$

$$+ 2 \sum_{i=3}^{s} \chi_i \sum_{j=2}^{i-1} \phi_j \alpha_{j,i-1} \left(q\chi_j w_j - \gamma_j\right) \tag{29}$$

At the nth step, compute H^n by substituting the current values ϕ^n , p^n , χ_i^n , γ_i^n , w_i^n , α_{kj}^n for ϕ, p , χ_i , γ_i , w_i , α_{kj} , respectively, in formulas (27), (28), (29). Also, compute $V^n = (H^n)^{-1}$, the inverse of the symmetric matrix H^n .

Printing Model D estimates $\hat{\phi}$, $\hat{p}$

After convergence, $\underset{\sim}{\theta}$ contains the Model D estimates $\hat{\phi}$, $\hat{p}$ and V the variance-covariance matrix.

That is,

$$\hat{\phi} = \theta_1 \quad , \qquad \mathrm{Var}\ \hat{\phi} = V_{11}$$

$$\hat{p} = \theta_2 \quad , \qquad \mathrm{Var}\ \hat{p} = V_{22}$$

$$\mathrm{Cov}(\hat{\phi},\hat{p}) = V_{12}$$

Print estimates, standard errors and covariance.

Model D estimates of M_i, N_i, B_i, variances and covariances

$$\hat{M}_i = \frac{m_i+z_i}{1-\hat{q}\hat{X}_i} \qquad i=2,\ldots,s \tag{30}$$

$$\left.\begin{aligned} \mathcal{m}_2 &= \hat{\phi}_1 R_1 \\ \mathcal{m}_{i+1} &= \hat{\phi}_i\left\{\mathcal{m}_i - \hat{p}\,\frac{d_i}{m_i}\,\mathcal{m}_i + R_i.\right\} \qquad i=2,\ldots,s-1 \end{aligned}\right\} \tag{31}$$

$$\hat{U}_i = \frac{u_i}{\hat{p}} \qquad i=2,\ldots,s \tag{32}$$

$$\hat{N}_i = \hat{U}_i+\hat{M}_i \qquad i=2,\ldots,s \tag{33}$$

$$\hat{B}_i = \hat{U}_{i+1} - \hat{\phi}_i\hat{q}\hat{U}_i \qquad i=2,\ldots,s-1 \tag{34}$$

To obtain variances/covariances for $\hat{M}_i$, $\hat{N}_i$, $\hat{B}_i$, first compute matrices X (2 by s-1) and W (2 by s-1) = VX.

Row 1 of X

$$X_{1,j-1} = \frac{q\chi_j}{\phi}\left[\sum_{k=2}^{j} t_{k-1}\,\alpha_{k-1,j-1}\left\{M_k + (1-q\chi_k)w_k\right\}\right.$$

$$\left. + \left(\frac{M_j}{1-q\chi_j} + w_j\right)\sum_{k=j+1}^{s} t_{k-1}\,\alpha_{j-1,k-1}\,(1-q\chi_k)\right]$$

$$j=2,\ldots,s \quad \text{or} \quad j-1=1,\ldots,s-1 \tag{35}$$

Row 2 of X

$$X_{21} = q\chi_2\left(\frac{M_2}{1-q\chi_2} + w_2\right)\sum_{k=2}^{s} \chi_k\,\alpha_{1,k-1}$$

$$X_{2,j-1} = q\chi_j \sum_{k=2}^{j-1} \alpha_{k,j-1}\phi_k\left[q\chi_k w_k - M_k\right]$$

$$+ q\chi_j\left(\frac{M_j}{1-q\chi_j} + w_j\right)\sum_{k=j}^{s} \chi_k\,\alpha_{j-1,k-1} \tag{36}$$

for $j=3,\ldots,s$ or $j-1=2,\ldots,s-1$.

Compute $W = VX$ by matrix multiplication, and the variances below which involve the elements of W and V, and S_i , P_i defined below.

$$\text{Compute} \quad S_i = \begin{cases} \frac{q}{\phi} \sum_{k=i}^{s-1} t_k \, \alpha_{i-1,k-1}(1-\chi_k) & i=2,\ldots,s-1 \\ 0 & i=s \end{cases} \tag{37}$$

$$P_i = \begin{cases} \chi_i + q\phi_i \sum_{k=i}^{s-1} \alpha_{ik} \, \chi_{k+1} & i=2,\ldots,s-1 \\ 1 & i=s \end{cases} \tag{38}$$

$$\begin{aligned} \mathrm{Var}(\hat{M}_i) = \frac{M_i}{1-q\chi_i} \Big\{ & q\chi_i + \frac{V_{11}M_i}{1-q\chi_i} S_i^2 + V_{22} \frac{M_i}{1-q\chi_i} P_i^2 \\ & -2S_iW_{1,i-1} - 2P_iW_{2,i-1} + 2S_iP_iV_{12} \frac{M_i}{1-q\chi_i} \Big\} \end{aligned}$$

$$i=2,\ldots,s \tag{39}$$

$$\mathrm{Cov}(\hat{M}_i,\hat{M}_j) = \frac{q\chi_j M_i}{1-q\chi_i}\,\alpha_{i-1,j-1} - \frac{M_j}{1-q\chi_j}\left[S_j W_{1,i-1} + P_j W_{2,i-1}\right]$$

$$- \frac{M_i S_i}{1-q\chi_i}\left[W_{1,j-1} - \frac{S_j M_j}{1-q\chi_j}\,V_{11} - \frac{P_j M_j}{1-q\chi_j}\,V_{12}\right]$$

$$- \frac{M_i P_i}{1-q\chi_i}\left[W_{2,j-1} - \frac{S_j M_j}{1-q\chi_j}\,V_{12} - \frac{P_j M_j}{1-q\chi_j}\,V_{22}\right]$$

$$i<j,\ i=2,\ldots,s-1,\ j=i+1,\ldots,s \qquad (40)$$

[Note that (40) with $i=j$ should give the same numerical result as (39) for the same value of i.]

$$\mathrm{Cov}(\hat{p},\hat{M}_i) = W_{2,i-1} - \frac{M_i S_i}{1-q\chi_i}\,V_{12} - \frac{M_i P_i}{1-q\chi_i}\,V_{22} \qquad i=2,\ldots,s \qquad (41)$$

The variances and covariances in (39), (40), (41), and in V are used to compute $\mathrm{Var}(\hat{N}_i)$, $\mathrm{Var}(\hat{B}_i)$, and covariances.

$$\mathrm{Var}(\hat{N}_i) = \mathrm{Var}(\hat{M}_i) + \frac{U_i}{p}\left\{q + \frac{U_i}{p}\,V_{22} - 2\mathrm{Cov}(\hat{p},\hat{M}_i)\right\} \qquad (42)$$

$$i=2,\ldots,s$$

$$\mathrm{Cov}(\hat{N}_i,\hat{N}_j) = \mathrm{Cov}(\hat{M}_i,\hat{M}_j) - \frac{U_j}{p}\,\mathrm{Cov}(\hat{p},\hat{M}_i) - \frac{U_i}{p}\,\mathrm{Cov}(\hat{p},\hat{M}_j)$$

$$+ \frac{U_i U_j}{p^2}\,V_{22} \qquad i=2,\ldots,s-1,\ j=i+1,\ldots,s \qquad (43)$$

$$\mathrm{Var}(\hat{B}_i) = \frac{qU_{i+1}}{p} + q\phi_i U_i \left(1 + \frac{q\phi_i}{p}\right) + \left(\frac{U_{i+1}-\phi_i U_i}{p}\right)^2 V_{22}$$

$$+ \left(\frac{t_i q\phi_i U_i}{\phi}\right)^2 V_{11} + 2\,\frac{U_{i+1}-\phi_i U_i}{p}\,\frac{t_i q\phi_i U_i}{\phi}\,V_{12}$$

$$i=2,\ldots,s-1 \qquad (44)$$

$\begin{pmatrix}\text{OMIT if}\\ s=3\end{pmatrix}$

$$\mathrm{Cov}(\hat{B}_{i-1},\hat{B}_i) = -\frac{\phi_i qU_i}{p} + V_{22}\left(\frac{U_i-\phi_{i-1}U_{i-1}}{p}\right)\left(\frac{U_{i+1}-\phi_i U_i}{p}\right)$$

$$+ V_{11}\left(\frac{t_{i-1}q\phi_{i-1}U_{i-1}}{\phi}\right)\left(\frac{t_i q\phi_i U_i}{\phi}\right)$$

$$+ V_{12}\left[\frac{t_{i-1}q\phi_{i-1}U_{i-1}}{\phi}\,\frac{U_{i+1}-\phi_i U_i}{p} + \frac{t_i q\phi_i U_i}{\phi}\,\frac{U_i-\phi_{i-1}U_{i-1}}{p}\right]$$

$$i=3,\ldots,s-1 \qquad (45)$$

$\begin{pmatrix}\text{OMIT if}\\ s=3 \text{ or } 4\end{pmatrix}$

$$\mathrm{Cov}(\hat{B}_i,\hat{B}_j) = V_{22}\left(\frac{U_{i+1}-\phi_i U_i}{p}\right)\left(\frac{U_{j+1}-\phi_j U_j}{p}\right) + V_{11}\,\frac{t_i q\phi_i U_i}{\phi}\,\frac{t_j q\phi_j U_j}{\phi}$$

$$+ V_{12}\left[\frac{t_i q\phi_i U_i}{\phi}\,\frac{U_{j+1}-\phi_j U_j}{p} + \frac{t_j q\phi_j U_j}{\phi}\,\frac{U_{i+1}-\phi_i U_i}{p}\right]$$

$$i<j-1,\ i=2,\ldots,s-3,\ j=i+2,\ldots,s-1 \qquad (46)$$

TESTING BETWEEN MODELS A, B AND D.

There are several test statistics which could be used. Jolly (1982) presents two statistics [equations (47), (48)] which are based on the proportions $\frac{r_i}{R_i}$, $i=1,\ldots,s-1$. However, the proportions $\frac{m_i}{m_i+z_i}$ also provide information about fit of the models, and this information is not utilized in Jolly's test statistics. I would like to compute and print separately the components corresponding to these two proportions to see how much difference the second component makes, especially for the data in Jolly's example. For the general user, however, only the sum of the two components need be printed.

Jolly presents two different types of test statistics, one for comparing B with A (eqn. 47), the other for comparing D with B (eqn. 48). The latter is based on the likelihood ratio; the former is based more directly on a chi-square statistic. The two are equivalent in large samples, and there is no overwhelming statistical argument supporting the use of one type rather than the other. We can either compute the test statistics as Jolly does (<u>but</u> with the second component included), or we can compute a likelihood ratio test statistic in both cases.

The instructions for programming these tests are therefore preliminary. They will enable us to compare results with those of Jolly. Later it will be necessary to make some changes to produce a version for general use.

Test of Model D vs Model B

Notation $\hat{\chi}_{iB}$ and $\hat{\chi}_{iD}$ are the model B and model D estimates of χ_i, respectively, $i=1,\ldots,s-1$.

$\hat{\rho}_{iB} = \dfrac{\hat{p}_{iB}}{1-\hat{q}_{iB}\hat{\chi}_{iB}}$ and $\hat{\rho}_{iD} = \dfrac{\hat{p}_D}{1-\hat{q}_D\hat{\chi}_{iD}}$ are the model B and model D estimates, respectively, of $\dfrac{p_i}{1-q_i\chi_i}$, $i=2,\ldots,s-1$.

Compute

$$L_1 = -2\sum_{i=1}^{s-1}\left\{r_i \log_e\left(\frac{1-\hat{\chi}_{iD}}{1-\hat{\chi}_{iB}}\right) + \left(R_i - r_i\right) \log_e\left(\frac{\hat{\chi}_{iD}}{\hat{\chi}_{iB}}\right)\right\} \tag{47}$$

$$L_2 = -2\sum_{i=2}^{s-1}\left\{m_i \log_e\left(\frac{\hat{\rho}_{iD}}{\hat{\rho}_{iB}}\right) + z_i \log_e\left(\frac{1-\hat{\rho}_{iD}}{1-\hat{\rho}_{iB}}\right)\right\} \tag{48}$$

Compute $L_1 + L_2$

Print out L_1, L_2

Print out "Total chisquare =" L_1+L_2

"Degrees of freedom =" $s-2$

"Probability =" (computed in usual way).

Test of Model D vs Model A

Compute two test statistics, print out both.

(i)
$$L_1 = -2 \sum_{i=1}^{s-1} \left\{ r_i \log_e \left[\frac{R_i(1-\hat{\chi}_{iD})}{r_i} \right] + \left(R_i - r_i\right) \log_e \left[\frac{R_i \hat{\chi}_{iD}}{R_i - r_i} \right] \right\} \tag{49}$$

$$L_2 = -2 \sum_{i=2}^{s-1} \left\{ m_i \log_e \left[\frac{(m_i+z_i)\hat{\rho}_{iD}}{m_i} \right] + z_i \log_e \left[\frac{(m_i+z_i)(1-\hat{\rho}_{iD})}{z_i} \right] \right\} \tag{50}$$

Print out L_1, L_2

and "Total chi square" $= L_1+L_2$

"Degrees of freedom" $= 2s-5$

"Probability" = (computed in usual way).

(ii) Compute
$$T_{1i} = \frac{[r_i - R_i(1-\hat{\chi}_{iD})]^2}{R_i \hat{\chi}_{iD}(1-\hat{\chi}_{iD})} \qquad i=1,\ldots,s-1 \tag{51}$$

$$T_{2i} = \frac{[m_i - (m_i+z_i)\hat{\rho}_{iD}]^2}{(m_i+z_i)\hat{\rho}_{iD}(1-\hat{\rho}_{iD})} \qquad i=2,\ldots,s-1 \tag{52}$$

Print out individual chi-square values $T_{11},\ldots,T_{1,s-1}$

and $T_{22},\ldots,T_{2,s-1}$.

Also print out
$$T_1 = \sum_{i=1}^{s-1} T_{1i} \quad , \qquad T_2 = \sum_{i=2}^{s-1} T_{2i}$$

and "Total chi square" $= T_1+T_2$

"Degrees of freedom" $= 2s-5$

"Probability" = (computed in usual way).

Test of Model B vs Model A (Omit if s=3)

Compute two test statistics as for the test of D vs A .

(i) Compute L_1 as in (49) but with $\hat{\chi}_{iB}$ in place of $\hat{\chi}_{iD}$.

Compute L_2 as in (50) but with $\hat{\rho}_{iB}$ in place of $\hat{\rho}_{iD}$.

Print out L_1, L_2 and (in the usual format) the total chi-square $= L_1+L_2$, with degrees of freedom = s-3, and probability.

(ii) Compute T_{1i} as in (51) but with $\hat{\chi}_{iB}$ in place of $\hat{\chi}_{iD}$.

Compute T_{2i} as in (52) but with $\hat{\rho}_{iB}$ in place of $\hat{\rho}_{iD}$.

Compute $T_1 = \sum_{1}^{s-1} T_{1i}$, $T_2 = \sum_{2}^{s-1} T_{2i}$.

Print out individual chi-square values, $T_{11}, \ldots, T_{1,s-1}$, $T_{22}, \ldots, T_{2,s-1}$; also, T_1, T_2 and total chi-square $= T_1+T_2$ with degrees of freedom = s-3.

Comments on Structure of Program

The order in which computations are carried out for models A (i.e., the Jolly-Seber model), B and D should depend on the data set to be analyzed. For "good" data sets, the best way to proceed is to do the computations for model A first, and use the Jolly-Seber or model A estimates $\hat{\phi}_i$, $\hat{p}_i$ to get starting values for the model B algorithm. For poor data sets where some summary statistics are zero, it may be better to start with the simplest model (model D) then proceed to B then A. This leads to the following possibilities:

OPTION 1: (for good data sets--should be the default?)

(i) First compute Jolly-Seber estimates (model A estimates).

(ii) Use these to get starting values for the model B algorithm (see the model B instructions).

(iii) Use model B estimates $\hat{\phi}$, $\hat{p}_i$ to get starting values for the model D algorithm (see the model D instructions).

(iv) Proceed to tests.

OPTION 2: (for poor data sets)

(i) Begin with the model D algorithm using starting values ϕ^0, p^0 read in with the data.

(ii) Use the model D estimates $\hat{\phi}$, $\hat{p}$ to get starting values for model B; (i.e., $\hat{\phi}$ from model D is passed as the initial value, ϕ^0 , and $\hat{p}$ from model D is passed as p_i^0, $i=2,\ldots,s$, to the model B algorithm). This is instead of reading in ϕ^0 and p_i^0 as initial values for Model B as I had suggested earlier.

(iii) Proceed to Jolly-Seber (model A) computations and tests, if possible.

Part 2

COMPUTER ALGORITHMS FOR MODELS B2 AND D2
To be incorporated in program JOLLYAGE

Reduced parameter models for the two-age class case.

With age-dependent models, the time for an animal to mature from the first to the second age-class must be the same as (or simply related to) the period between successive bandings. Thus, as in Pollock (1981), the period between bandings and the time spent in age-class 0, are both assumed to be one year in the models considered here. That is, $t_i=1$ for $i=1,\dots,s-1$, so that for the models with constant survival, $\varphi_i=\varphi^{t_i}=\varphi$, $i=1,\dots,s-1$. I cannot think of a useful way to generalize and allow variable t_i in the two-age-class models.

Outline of models considered

Model A2 - variable or time-specific survival for adults and young, variable capture rates.

- age dependent generalization of Jolly-Seber Model
- same structure as Pollock's (1981) model, but with M_i viewed as variables, not fixed parameters.
- estimable survival and capture rate parameters are

$$\varphi_1^a,\dots,\varphi_{s-2}^a,\ \varphi_1^y,\dots,\varphi_{s-2}^y,\ p_2,\dots,p_{s-1},\ \varphi_{s-1}^a\, p_s\ ,\ \varphi_{s-1}^y\, p_s\ .$$

Model B2.

- constant survival for adults and young, variable capture rates.
- age-dependent generalization of Jolly's (1982) Model B.
- estimable survival and capture rate parameters are φ^a, φ^y, $p_2,\ldots,p_s$.

Model D2 - constant survival for adults and young, constant capture rates.

- age-dependent generalization of Jolly's (1982) Model D.
- estimable survival and capture rate parameters are φ^a, φ^y, p.

Notation: Because of the complexity of formulae below, it seemed less confusing to use superscripts "a" and "y", instead of a prime or "0" or "1" to denote age dependence. The relationship between notation here, that in Pollock (1981), and the JOLLYAGE output is indicated below (but note that I have a question concerning the equivalence of NB(I) and z_i).

Program JOLLYAGE

Pollock (1981)	(page 1 of output)	Here
n_i^1	NN(I)	n_i^a
n_i^0	NN´(I)	$n_i^y = u_i^y$
m_i^1	NM(I)	m_i
z_i^1	NB(I) ? ?	z_i
R_i^1	S(I)	R_i^a
R_i^0	S´(I)	R_i^y
r_i^1	R(I)	r_i^a
r_i^0	R´(I)	r_i^y
N_i^1	N(I)	N_i^a
$M_i^1 + M_i^2$	M(I)	M_i
φ_i^1	PHI(I)	φ_i^a
φ_i^0	PHI´(I)	φ_i^y
	P(I)	p_i

Note: To clarify definition of z_i as used here, $z_1=0$, $z_2=r_i^a+r_i^y-m_2$ and $z_{i+1} = z_i+r_i^a+r_i^y-m_{i+1}$, $i=2,\ldots,s-2$, $z_s=0$.

Additional notation

$\chi_s^a = 1$ (where s is the number of samples)

$1 - \chi_i^a = \varphi^a\left(1-q_{i+1}\ \chi_{i+1}^a\right)$ $\quad i=1,\dots,s-1$

$1 - \chi_i^y = \varphi^y\left(1-q_{i+1}\ \chi_{i+1}^a\right)$ $\quad i=1,\dots,s-1$

u_i^a = no. of unmarked adults caught at i .

U_i^a = no. of unmarked adults present just before sample i .

Define $M_1=0$, $m_1=0$, $z_1=0$, and $z_s=0$.

Input for Model B2 algorithm

R_i^y , R_i^a , m_i , z_i , r_i^y , r_i^a , s , u_i^a

plus starting values for φ^a, φ^y, $p_2,\dots,p_s$.

<u>OPTION 1</u>: These starting values can be computed by averaging the Pollock (1981) estimates $\hat{\varphi}_i^1$ and $\hat{\varphi}_i^0$, and from

$$\hat{p}_i = \frac{m_i}{m_i + R_i^a \dfrac{z_i}{r_i^a}} \quad , \qquad i=2,\dots,s-1 \quad .$$

<u>OPTION 2</u>: Starting values for φ^a, φ^y, $p_2,\dots,p_s$ can be obtained from the Model D2 estimates of constant φ^a, φ^y and p (to be described later).

Computations

The iterative procedure is carried out to produce ML estimates $\hat{\varphi}^a$, $\hat{\chi}_1^a$,... $\hat{\chi}_{s-1}^a$, $\hat{\varphi}^y$. From these estimates,

$$\hat{p}_i = \frac{1}{\hat{\chi}_i^a}\left[\frac{1-\hat{\chi}_{i-1}^a}{\hat{\varphi}^a} - \left(1-\hat{\chi}_i^a\right)\right] ,$$ is obtained, $i=2,\ldots,s$.

Given starting values $\varphi^{a,0}$, $\varphi^{y,0}$, $p_2^0,\ldots,p_s^0$, calculate

$q_i^0 = 1-p_i^0$, $i=2,\ldots,s$, and then obtain

starting values for $1-\chi_i^a$, and $1-\chi_i^y$, $i=1,\ldots,s-1$

using the formulas

$$1-\chi_i^a = \varphi^a\left(1-q_{i+1}\ \chi_{i+1}^a\right) ,$$

$$1-\chi_i^y = \varphi^y\left(1-q_{i+1}\ \chi_{i+1}^a\right) ,$$

where $\chi_s^a \equiv 1$.

Iteration is carried out as for Model B in program JOLLY, but with θ , Δ , H and V as defined below.

The vector of estimates $\underset{(s+1)\times 1}{\theta}$ contains the elements

$$\underset{(s+1)\times 1}{\tilde{\theta}} = \begin{bmatrix} \hat{\varphi}^a \\ \hat{\chi}_1^a \\ \hat{\chi}_2^a \\ | \\ | \\ | \\ \hat{\chi}_{s-1}^a \\ \hat{\varphi}^y \end{bmatrix} .$$

At each step of the iteration, elements in $\underset{\sim}{\theta}$ are used to calculate updated values for p_i, q_i, i=2,...,s, and $1-\chi_i^y$, i=1,...,s-1. These are used to re-evaluate $\underset{(s-1)\times 1}{\Delta}$, $\underset{(s+1)\times(s+1)}{H}$ and $\underset{(s+1)\times(s+1)}{V} = H^{-1}$ using the formulas below.

<u>**Elements of $\underset{\sim}{\Delta}$**</u>

$$\Delta_1 = \frac{1}{\varphi^a} \sum_{i=2}^{s-1} \frac{1}{\chi_i^a} \left\{ \frac{z_i}{q_i} - \frac{m_i(1-\chi_i^a)}{p_i} \right\} \tag{1}$$

$$\Delta_2 = \frac{R_1^y - r_1^y}{\chi_1^y} \frac{\varphi^y}{\varphi^a} + \frac{R_1^a - r_1^a}{\chi_1^a} - \frac{1}{\varphi^a \chi_2^a} \left[\frac{m_2}{p_2} - \frac{z_2}{q_2} \right] \tag{2}$$

$$\Delta_{i+1} = \frac{R_i^y - r_i^y}{\chi_i^y} \frac{\varphi^y}{\varphi^a} + \frac{1}{\chi_i^a} \left(\frac{m_i q_i}{p_i} - z_i + R_i^a - r_i^a \right)$$

$$- \frac{1}{\varphi^a \chi_{i+1}^a} \left[\frac{m_{i+1}}{p_{i+1}} - \frac{z_{i+1}}{q_{i+1}} \right] . \tag{3}$$

i+1=3,...,s

$$\Delta_{s+1} = \frac{1}{\varphi^y} \sum_{i=1}^{s-1} \left[r_i^y - \frac{(1-\chi_i^y)}{\chi_i^y} \left(R_i^y - r_i^y \right) \right] . \tag{4}$$

Elements of H

Row 1

$$H_{11} = \frac{1}{(\varphi^a)^2} \sum_{i=2}^{s-1} \left[\frac{R_i^y(1-\chi_i^y)}{\chi_i^y} + \frac{1-q_i\chi_i^a}{(\chi_i^a)^2} \left\{ \frac{m_i(1-\chi_i^a)}{p_i^2} + \frac{z_i}{q_i^2} \right\} \right] + \frac{R_1^y(1-\chi_1^y)}{(\varphi^a)^2 \; \chi_1^y} \tag{5}$$

$$H_{12} = \frac{1}{(\varphi^a)^2} \left[\frac{\varphi^y \; R_1^y}{\chi_1^y} + \frac{1}{(\chi_2^a)^2} \left\{ \frac{m_2(1-\chi_2^a)}{p_2^2} + \frac{z_2}{q_2^2} \right\} \right] \tag{6}$$

$$H_{1,i+1} \quad \frac{1}{\varphi^a} \left[\frac{\varphi^y}{\varphi^a} \frac{R_i^y}{\chi_i^y} - \frac{(1-q_i \; \chi_i^a) m_i}{(p_i \; \chi_i^a)^2} + \frac{1}{\varphi^a (\chi_{i+1}^a)^2} \left\{ \frac{m_{i+1}(1-\chi_{i+1}^a)}{p_{i+1}^2} + \frac{z_{i+1}}{q_{i+1}^2} \right\} \right] \tag{7}$$

$i+1=3,\ldots,s$

$$H_{1,s+1} = -\frac{1}{\varphi^a \varphi^y} \sum_{i=1}^{s-1} \frac{(1-\chi_i^y) R_i^y}{\chi_i^y} \tag{8}$$

Rows 2 to 5

$$H_{22} = \left(\frac{\varphi^y}{\varphi^a}\right)^2 \frac{R_1^y}{\chi_1^y} + \frac{R_1^a}{\chi_1^a} + \frac{1}{\left(\varphi^a\chi_2^a\right)^2}\left(\frac{m_2}{p_2^2} + \frac{z_2}{q_2^2}\right) \tag{9}$$

$$H_{i+1,i+1} = \left(\frac{\varphi^y}{\varphi^a}\right)^2 \frac{R_i^y}{\chi_i^y} + \frac{R_i^a}{\chi_i^a} + \frac{1}{\left(\chi_i^a\right)^2}\left[\frac{m_i}{p_i^2} - m_i - z_i\right]$$

$$+ \frac{1}{\left(\varphi^a\chi_{i+1}^a\right)^2}\left[\frac{m_{i+1}}{p_{i+1}^2} + \frac{z_{i+1}}{q_{i+1}^2}\right] \tag{10}$$

$i+1=3,\ldots,s$

$$H_{i,i+1} = -\frac{m_i}{\varphi^a\left(p_i\chi_i^a\right)^2} \qquad i=2,\ldots,s-1 \tag{11}$$

$$H_{ij} = 0 \qquad i=2,\ldots,s-1\ ,\ i+1<j\leq s \tag{12}$$

$$H_{i,s+1} = -\frac{R_{i-1}^y}{\varphi^a\chi_{i-1}^y} \qquad i=2,\ldots s \tag{13}$$

Row s+1

$$H_{s+1,s+1} = \frac{1}{\left(\varphi^y\right)^2}\sum_{i=1}^{s-1}\frac{\left(1-\chi_i^y\right)R_i^y}{\chi_i^y} \tag{14}$$

Elements H_{ij} for which $i>j$ are obtained from the above by symmetry,

i.e., $H_{ij}=H_{ji}$, $\quad i=2,\ldots,s$, $\; j=1,\ldots,i-1$.

Model B2 estimates

After convergence, the elements of θ are

$$\theta_1=\hat{\varphi}^a , \qquad \theta_{s+1}=\hat{\varphi}^y , \quad \text{and} \quad \theta_i=\hat{\chi}^a_{i-1} , \qquad i=2,\ldots,s .$$

$$\text{and} \quad V_{11} = \mathrm{var}(\hat{\varphi}^a) , \quad V_{s+1,s+1} = \mathrm{var}(\hat{\varphi}^y) ,$$

$$V_{1,s+1} = \mathrm{Cov}(\hat{\varphi}_a,\hat{\varphi}_y) , \quad \text{etc.}$$

It may not be necessary to print out $\hat{\chi}^a_i$ (or $1-\hat{\chi}^a_i$) and variances/covariances. However, $\hat{p}_i$ and variances/covariances should be calculated (as below) and printed.

$$\hat{p}_i = \frac{1}{\hat{\chi}^a_i}\left[\frac{1-\hat{\chi}^a_{i-1}}{\hat{\varphi}^a} - (1-\hat{\chi}^a_i)\right] , \qquad 1=2,\ldots,s-1 \qquad (15)$$

$$\begin{aligned}\mathrm{Cov}(\hat{p}_i,p_j) = \frac{1}{\chi^a_i\,\chi^a_j}\Bigg\{&\frac{(1-q_i\,\chi^a_i)}{\varphi^a}\left[\frac{(1-q_j\,\chi^a_j)}{\varphi^a}\;V_{11} + \frac{V_{1j}}{\varphi^a} - q_j\;V_{1,j+1}\right]\\ &+ \frac{1}{\varphi^a}\left[\frac{(1-q_j\,\chi^a_j)}{\varphi^a}\;V_{1i} + \frac{V_{ij}}{\varphi^a} - q_j\;V_{i,j+1}\right]\\ &- q_i\left[\frac{(1-q_j\,\chi^a_j)}{\varphi^a}\,V_{1,i+1} + \frac{V_{i+1,j}}{\varphi^a} - q_j\;V_{i+1,j+1}\right]\Bigg\}\end{aligned}$$

$$2\leq i,j<s \qquad (16)$$

$$\begin{aligned}\mathrm{Cov}(\hat{p}_i,\hat{p}_s) = \frac{1}{\chi^a_i}\Bigg\{&\frac{(1-q_i\,\chi^a_i)}{\varphi^a}\left[\frac{p_s}{\varphi^a}\,V_{11} + \frac{V_{1s}}{\varphi^a}\right]\\ &+ \frac{1}{\varphi^a}\left[\frac{p_s\;V_{1i}}{\varphi^a} + \frac{V_{is}}{\varphi^a}\right] - q_i\left[\frac{p_s\;V_{1,i+1}}{\varphi^a} + \frac{V_{i+1,s}}{\varphi^a}\right]\Bigg\}\end{aligned}$$

$$i=2,\ldots,s-1 \qquad (17)$$

$$\mathrm{Var}(\hat{p}_s) = \frac{V_{ss}}{(\varphi^a)^2} + \frac{p_s^2}{(\varphi^a)^2} V_{11} + \frac{2p_s}{(\varphi^a)^2} V_{1s} \tag{18}$$

$$\mathrm{Cov}(\hat{\varphi}^a, \hat{p}_i) = \begin{cases} \dfrac{1}{\chi_i^a}\left[q_i\, V_{1,i+1} - \dfrac{V_{1i}}{\varphi^a} - \dfrac{(1-q_i\, \chi_i^a)}{\varphi^a} V_{11}\right] & i=2,\ldots,s-1 \\ -\dfrac{1}{\varphi^a}\left[V_{1s} + p_s\, V_{11}\right] & i=s \end{cases} \tag{19}$$

$$\mathrm{Cov}(\hat{\varphi}^y, \hat{p}_i) = \begin{cases} \dfrac{1}{\chi_i^a}\left[q_i\, V_{i+1,s+1} - \dfrac{V_{i,s+1}}{\varphi^a} - \dfrac{(1-q_i\, \chi_i^a)}{\varphi^a} V_{1,s+1}\right] & i=2,\ldots,s-1 \\ -\dfrac{1}{\varphi^a}\left[V_{s,s+1} + p_s\, V_{1,s+1}\right] & i=s \end{cases} \tag{20}$$

Model B2 estimates of M_i , N_i^a , B_i^a .

Note that $M_i^y = 0$ by definition, M_i is actually M_i^a . Also, N_i^y , B_i^y are not estimable.

$$\hat{M}_i = \frac{m_i+z_i}{1-\hat{q}_i\, \hat{\chi}_i^a} \qquad i=2,\ldots,s$$

$$\hat{U}_i^a = \frac{u_i^a}{\hat{p}_i} \qquad i=2,\ldots,s$$

$$\hat{N}_i^a = \hat{U}_i^a + \hat{M}_i \qquad i=2,\ldots,s$$

$$\hat{B}_i^a = \hat{U}_{i+1}^a - \hat{\varphi}^a\, \hat{q}_i\, \hat{U}_i^a + R_i^y\, \hat{\varphi}^y \qquad i=2,\ldots,s-1$$

To obtain variances/covariances for $\hat{M}_i$, $\hat{N}_i^a$, $\hat{B}_i^a$, compute matrices X and W defined below.

$$\text{Let } \alpha_{ij} = \begin{cases} 1 & \text{if } i=j,\ j=1,\ldots,s-1 \\ q_{i+1}\varphi^a \ldots q_j \varphi^a & \text{if } i<j,\ j=2,\ldots,s-1, i=1,\ldots,j \end{cases} .$$

$$\text{Thus, } \alpha_{ij} = (\varphi^a)^{j-i} \prod_{k=i+1}^{j} q_k \qquad \text{if } i<j$$

<u>Elements of X (s+1) by (s-1)</u>

<u>Row 1</u>

$$X_{11} = -q_2\, \chi_2^a \frac{R_1^y}{\chi_1^y} \frac{\varphi^y}{\varphi^a}$$

$$X_{1j} = q_{j+1}\, \chi_{j+1}^a \left\{ \sum_{i=2}^{j} \frac{M_i \alpha_{ij}}{\chi_i^a} - \frac{\varphi^y}{\varphi^a} \sum_{i=1}^{j} \frac{R_i^y \alpha_{ij}}{\chi_i^y} \right\} \qquad j=2,\ldots,s-1$$

<u>Rows 2 to s</u>

$$X_{21} = -\frac{q_2\, \chi_2^a}{1-\chi_1^a} \left\{ \frac{R_1^a \varphi^a}{\chi_1^a} + \frac{R_1^y \varphi^y}{\chi_1^y} \right\}$$

$$X_{2j} = \frac{q_{j+1}\, \chi_{j+1}^a\, \alpha_{ij}}{\chi_2^a} \left\{ \frac{R_1^a \varphi^a (1-\varphi^a)}{\chi_1^a} + \frac{R_1^y \varphi^y (1-\varphi^y)}{\chi_1^y} \right\} \qquad j=2,\ldots,s-1$$

$$X_{i,i-1} = -\frac{q_i\, \chi_i^a}{1-\chi_{i-1}^a} \left\{ \frac{(M_{i-1}\, q_{i-1} + R_{i-1}^a)\varphi^a}{\chi_{i-1}^a} + \frac{R_{i-1}^y \varphi^y}{\chi_{i-1}^y} \right\} \qquad i=3,\ldots,s$$

$$X_{ij} = \frac{q_{j+1}\,\chi^a_{j+1}\,\alpha_{ij}}{\chi^a_i}\left\{\frac{\left(M_{i-1}\,q_{i-1} + R^a_{i-1}\right)\varphi^a\left(1-\varphi^a\right)}{\chi^a_{i-1}} + \frac{R^y_{i-1}\,\varphi^y\left(1-\varphi^y\right)}{\chi^y_{i-1}}\right\}$$

$$i=3,\ldots,s-1,\quad j=i,\ldots,s-1$$

$$X_{ij} = 0 \qquad j\leq i-2,\quad i=3,\ldots,s$$

Row s+1

$$X_{s+1,j} = q_{j+1}\,\chi^a_{j+1}\sum_{i=1}^{j}\frac{R^y_i\,\alpha_{ij}}{\chi^y_i} \qquad j=1,\ldots,s-1$$

Next compute the (s+1) by (s-1) matrix W by W=VX .

$$\mathrm{Var}\left(\hat{M}_i\right) = M_i\left\{\frac{q_i\,\chi^a_i}{1-q_i\chi^a_i} + \frac{M_i\,V_{11}}{\left(\varphi^a\right)^2} + \frac{M_i\,V_{ii}}{\left(1-\chi^a_{i-1}\right)^2} + \frac{2M_i}{\varphi^a\left(1-\chi^a_{i-1}\right)}\,V_{1i}\right.$$

$$\left. + \frac{2}{\varphi^a}\,W_{1,i-1} + \frac{2}{1-\chi^a_{i-1}}\,W_{i,i-1}\right\} \qquad i=2,\ldots,s$$

$$\mathrm{Cov}\left(\hat{M}_i,\hat{M}_j\right) = \frac{q_j\,\chi^a_j\,M_i}{1-q_i\,\chi^a_i}\,\alpha_{i-1,j-1} + \frac{M_i}{\varphi^a}\,W_{1,j-1} + \frac{M_i}{\left(1-\chi^a_{i-1}\right)}\,W_{i,j-1}$$

$$+ \frac{M_j}{\varphi^a}\left[\frac{M_i}{\varphi^a}\,V_{11} + W_{1,i-1} + \frac{M_i}{\left(1-\chi^a_{i-1}\right)}\,V_{1i}\right]$$

$$+ \frac{M_j}{1-\chi^a_{j-1}}\left[W_{j,i-1} + \frac{M_i}{\varphi^a}\,V_{1j} + \frac{M_i}{\left(1-\chi^a_{i-1}\right)}\,V_{ij}\right]$$

$$i<j,\quad i=2,\ldots,s-1,\quad j=i+1,\ldots,s$$

$$\mathrm{Cov}(\hat{p}_i,\hat{M}_j) = \frac{1}{\chi_i^a}\Bigg[q_i\, W_{i+1,j-1} - \frac{W_{i,j-1}}{\varphi^a} - \frac{(1-q_i\,\chi_i^a)}{\varphi^a}\, W_{1,j-1}$$

$$+ \frac{M_j}{\varphi^a}\left\{q_i\, V_{1,i+1} - \frac{V_{1i}}{\varphi^a} - \frac{(1-q_i\,\chi_i^a)}{\varphi^a}\, V_{11}\right\}$$

$$+ \frac{M_j}{1-\chi_{j-1}^a}\left\{q_i\, V_{i+1,j} - \frac{V_{ij}}{\varphi^a} - \frac{(1-q_i\,\chi_i^a)}{\varphi^a}\, V_{1j}\right\}\Bigg]$$

$i=2,\ldots,s-1, \quad j=2,\ldots,s$.

$$\mathrm{Cov}(\hat{p}_s,\hat{M}_j) = -\frac{W_{s,j-1}}{\varphi^a} - \frac{p_s}{\varphi^a}\, W_{1,j-1}$$

$$-\frac{M_j}{\varphi^a}\left\{\frac{p_s}{\varphi^a}\, V_{11} + \frac{V_{1s}}{\varphi^a}\right\} - \frac{M_j}{1-\chi_{j-1}^a}\left\{\frac{p_s}{\varphi^a}\, V_{1j} + \frac{V_{sj}}{\varphi^a}\right\}$$

$j=2,\ldots,s$.

$$\mathrm{Var}(\hat{N}_i^a) = \mathrm{Var}(\hat{M}_i) + \frac{U_i^a}{p_i}\left\{q_i + \frac{U_i^a}{p_i}\ \mathrm{Var}(\hat{p}_i) - 2\mathrm{Cov}(\hat{p}_i,\hat{M}_i)\right\}$$

$i=2.,,,.s$

$$\mathrm{Cov}(\hat{N}_i^a,\hat{N}_j^a) = \mathrm{Cov}(\hat{M}_i,\hat{M}_j) - \frac{U_i^a}{p_i}\,\mathrm{Cov}(\hat{p}_i,\hat{M}_j) - \frac{U_j^a}{p_j}\,\mathrm{Cov}(\hat{p}_j,\hat{M}_i)$$

$$+ \frac{U_i^a}{p_j}\,\frac{U_j^a}{p_j}\,\mathrm{Cov}(\hat{p}_i,\hat{p}_j)$$

$i=2,\ldots,s-1; \quad j=i+1\ldots.,s$

$$\mathrm{Var}\left(\hat{B}_i^a\right) = \frac{U_{i+1}^a\ q_{i+1}}{p_{i+1}} + q_i \varphi^a U_i^a \left[1 + \frac{q_i \varphi^a}{p_i}\right]$$

$$+ \left(q_i\ U_i^a\right)^2\ V_{11} + \left(\frac{\varphi^a U_i^a}{p_i}\right)^2\ \mathrm{Var}\left(\hat{p}_i\right)$$

$$+ \left(\frac{U_{i+1}^a}{p_{i+1}}\right)^2\ \mathrm{Var}\left(\hat{p}_{i+1}\right) - \frac{2q_i \varphi^a \left(U_i^a\right)^2}{p_i}\ \mathrm{Cov}\left(\hat{\varphi}^a, \hat{p}_i\right)$$

$$+ \frac{2q_i U_i^a U_{i+1}^a}{p_{i+1}}\ \mathrm{Cov}\left(\hat{\varphi}^a, \hat{p}_{i+1}\right) - \frac{2\varphi^a U_i^a U_{i+1}^a}{p_i p_{i+1}}\ \mathrm{Cov}\left(\hat{p}_i, \hat{p}_{i+1}\right)$$

$$+ \left(R_i^y\right)^2\ V_{s+1,s+1} + 2R_i^y \left\{\varphi^a\ \frac{U_i^a}{p_i}\ \mathrm{Cov}\left(\hat{p}_i, \hat{\varphi}^y\right)\right.$$

$$\left. - U_i^a\ q_i\ V_{1,s+1} - \frac{U_{i+1}^a}{p_{i+1}}\ \mathrm{Cov}\left(\hat{p}_{i+1}, \hat{\varphi}^y\right)\right\}$$

$$i=2,\ldots,s-1$$

$\left[\begin{matrix}\text{OMIT if}\\ s=3\end{matrix}\right]$

$$\mathrm{Cov}\left(\hat{B}_{i-1}^a, \hat{B}_i^a\right) = -\frac{q_i \varphi^a U_i^a}{p_i} + q_{i-1}\ U_{i-1}^a \left\{q_i\ U_i^a\ V_{11}\right.$$

$$\left. - \frac{\varphi^a U_i^a}{p_i}\ \mathrm{Cov}\left(\hat{\varphi}^a, \hat{p}_i\right) + \frac{U_{i+1}^a}{p_{i+1}}\ \mathrm{Cov}\left(\hat{\varphi}^a, \hat{p}_{i+1}\right)\right\}$$

$$+ \frac{\varphi^a U_{i-1}^a}{p_{i-1}} \left\{-q_i U_i^a\ \mathrm{Cov}\left(\hat{\varphi}^a, \hat{p}_{i-1}\right)\right.$$

$$\left. + \frac{\varphi^a U_i^a}{p_i}\ \mathrm{Cov}\left(\hat{p}_{i-1}, \hat{p}_i\right) - \frac{U_{i+1}^a}{p_{i+1}}\ \mathrm{Cov}\left(\hat{p}_{i-1}, \hat{p}_{i+1}\right)\right\}$$

$$+ \frac{U_i^a}{p_i} \left\{q_i U_i^a\ \mathrm{Cov}\left(\hat{\varphi}^a, \hat{p}_i\right) - \frac{\varphi^a U_i^a}{p_i}\ \mathrm{Var}\left(\hat{p}_i\right)\right.$$

$$+ \frac{U_{i+1}^{a}}{p_{i+1}} \operatorname{Cov}\left(\hat{p}_i, \hat{p}_{i+1}\right)\Bigg\} + R_{i-1}^{y} R_{i}^{y} V_{s+1,s+1}$$

$$+ R_i^y \left\{ \varphi^a \frac{U_{i-1}^{a}}{p_{i-1}} \operatorname{Cov}\left(\hat{p}_{i-1}, \hat{\varphi}\right) - U_{i-1}^{a} q_{i-1} V_{1,s+1}\right.$$

$$\left. - \frac{U_i^a}{p_i} \operatorname{Cov}\left(\hat{p}_i, \hat{\varphi}^y\right) \right\}$$

$$+ R_{i-1}^{y} \left\{ \varphi^a \frac{U_i^a}{p_i} \operatorname{Cov}\left(\hat{p}_i, \hat{\varphi}^y\right) - U_i^a q_i V_{1,s+1}\right.$$

$$\left. - \frac{U_{i+1}^{a}}{p_{i+1}} \operatorname{Cov}\left(\hat{p}_{i+1}, \hat{\varphi}^y\right) \right\} \quad .$$

$i=3,\ldots,s-1$

$$\begin{bmatrix} \text{OMIT if} \\ s=3 \\ \text{or } 4 \end{bmatrix} \quad \operatorname{Cov}\left(\hat{B}_i^a, \hat{B}_j^a\right) = q_i U_i^a \left\{ q_j U_j^a V_{11} - \frac{\varphi^a U_j^a}{p_j} \operatorname{Cov}\left(\hat{\varphi}^a, \hat{p}_j\right)\right.$$

$$\left. + \frac{U_{j+1}^{a}}{p_{j+1}} \operatorname{Cov}\left(\hat{\varphi}^a, \hat{p}_{j+1}\right) \right\} + \frac{\varphi^a U_i^a}{p_i} \left\{ -q_j U_j^a \operatorname{Cov}\left(\hat{\varphi}^a, \hat{p}_i\right)\right.$$

$$\left. + \frac{\varphi^a U_j^a}{p_j} \operatorname{Cov}\left(\hat{p}_i, \hat{p}_j\right) - \frac{U_{j+1}^{a}}{p_{j+1}} \operatorname{Cov}\left(\hat{p}_i, \hat{p}_{j+1}\right) \right\}$$

$$+ \frac{U_{i+1}^{a}}{p_{i+1}} \left\{ q_j U_j^a \operatorname{Cov}\left(\hat{\varphi}^a, \hat{p}_{i+1}\right) \right.$$

$$- \frac{\varphi^a U_j^a}{p_j} \operatorname{Cov}(\hat{p}_{i+1}, \hat{p}_j) + \frac{U_{j+1}^a}{p_{j+1}} \operatorname{Cov}(\hat{p}_{i+1}, \hat{p}_{j+1}) \Bigg\}$$

$$+ R_i^y R_j^y V_{s+1,s+1}$$

$$+ R_i^y \left\{ \varphi^a \frac{U_j^a}{p_j} \operatorname{Cov}(\hat{p}_j, \varphi^y) - U_j^a q_j V_{1,s+1} \right.$$

$$\left. - \frac{U_{j+1}^a}{p_{j+1}} \operatorname{Cov}(\hat{p}_{j+1}, \hat{\varphi}^y) \right\}$$

$$+ R_j^y \left\{ \varphi^a \frac{U_i^a}{p_i} \operatorname{Cov}(\hat{p}_i, \hat{\varphi}^y) - U_i^a q_i V_{1,s+1} \right.$$

$$\left. - \frac{U_{i+1}^a}{p_{i+1}} \operatorname{Cov}(\hat{p}_{i+1}, \hat{\varphi}^y) \right\}$$

$$i=2,\ldots,s-3, \quad j=i+2,\ldots,s-1$$

Model D2

<u>Necessary Input</u>

Same as for Model B2, plus starting values for φ^a , φ^y and p , via one of several options.

OPTION 1.

Compute starting values for φ^a , φ^y and p by averaging the corresponding Model A2 estimates.

OPTION 2.

Use Model B2 estimates $\hat{\varphi}^a$, $\hat{\varphi}^y$, and an average of the $\hat{p}_i$, as starting values.

OPTION 3.

Read in starting values for φ^a , φ^y and p .

COMPUTATIONS

The iterative procedure produces ML estimates $\hat{\varphi}^a$, $\hat{\varphi}^y$ and $\hat{p}$, and the corresponding estimated variance covariance matrix V (dimension 3×3).

At each iteration, it is necessary to compute

$$q = 1-p$$

$$1-\chi_i^a = \varphi^a\left(1-q\ \chi_{i+1}^a\right) \qquad i=1,\ldots,s-1$$

(with $\chi_s^a \equiv 1$ as usual),

$$1-\chi_i^y = \varphi^y\left(1-q\ \chi_{i+1}^a\right) = \varphi^y\left(1-\chi_i^a\right)/\varphi^a \quad , \qquad i=1,\ldots,s-1,$$

$$\delta_i = \begin{cases} \sum_{k=i}^{s-1} \left(\varphi^a q\right)^{k-i} \left(1-\chi_k^a\right) & i=1,\ldots,s-1 \\ \left(= \varphi^a p \quad \text{if} \quad i=s-1\right) & \\ 0 & i=s \end{cases}$$

Iteration Scheme

Iteration proceeds as for Models B, D, B2, with $\underset{\sim}{\theta}$, $\underset{\sim}{\Delta}$, H and V as defined below.

Elements of $\underset{\sim}{\theta}$ (3×1).

$$\underset{\sim}{\theta} = \begin{pmatrix} \hat{\varphi}^a \\ \hat{p} \\ \hat{\varphi}^y \end{pmatrix}$$

Elements of $\underset{\approx}{\Delta}$ (3×1)

$$\Delta_1 = \frac{1}{\varphi^a} \sum_{i=1}^{s-1} \left\{ r_i^a + r_i^y + z_i - \delta_i \left(\frac{R_i^a - r_i^a}{\chi_i^a} + \frac{\varphi^y}{\varphi^a} \frac{R_i^y - r_i^y}{\chi_i^y} \right) \right\}$$

$$\Delta_2 = \frac{1}{pq} \sum_{i=1}^{s-1} \left\{ R_i^a - \frac{R_i^a - r_i^a}{\chi_i^a} \right\}$$

$$\Delta_3 = \frac{1}{\varphi^y} \sum_{i=1}^{s-1} \left\{ R_i^y - \frac{R_i^y - r_i^y}{\chi_i^y} \right\}$$

Elements of H (3×3)

$$H_{11} = \frac{1}{(\varphi^a)^2} \sum_{i=1}^{s-1} \left[\frac{\delta_i^2}{1-\chi_i^a} \left\{ \frac{R_i^a}{\chi_i^a} + \underbrace{\frac{m_i\, q}{1-q\, \chi_i^a}}_{\substack{\textit{Term vanishes}\\ \textit{for } i=1\\ \textit{as } m_1=0}} \right\} + \underbrace{\left(q\, \varphi^y\right)^2 \frac{\delta_{i+1}^2\, R_i^y}{\chi_i^y\left(1-\chi_i^y\right)}}_{\substack{\textit{Term vanishes}\\ \textit{for } i=s-1\\ \textit{as } \delta_s=0}} \right]$$

$$H_{22} = \sum_{i=1}^{s-1} \frac{1}{1-\chi_i^a} \left\{ \left(\frac{\delta_i}{q} - \frac{\left(1-\chi_i^a\right)}{pq} \right)^2 \left(\frac{R_i^a}{\chi_i^a} + \frac{\varphi^y}{\varphi^a} \frac{R_i^y}{\chi_i^y} \right) + \frac{\delta_i^2\, m_i}{q\left(1-q\, \chi_i^a\right)} \right\}$$

$$H_{33} = \frac{1}{(\varphi^y)^2} \sum_{i=1}^{s-1} \frac{R_i^y\left(1-\chi_i^y\right)}{\chi_i^y}$$

$$H_{12} = -\frac{1}{\varphi^a} \sum_{i=1}^{s-1} \frac{1}{1-\chi_i^a} \left\{ \left(\delta_i - \frac{\left(1-\chi_i^a\right)}{p} \right) \left(\frac{R_i^a}{\chi_i^a} \frac{\delta_i}{q} + \varphi^y\, \delta_{i+1} \frac{R_i^y}{\chi_i^y} \right) + \frac{m_i\, \delta_i^2}{1-q\, \chi_i^a} \right\}$$

$$H_{13} = \frac{q}{\varphi^a} \sum_{i=1}^{s-2} \frac{R_i^y}{\chi_i^y}\, \delta_{i+1}$$

$$H_{23} = -\frac{1}{q\, \varphi^a} \sum_{i=1}^{s-1} \frac{R_i^y}{\chi_i^y} \left(\delta_i - \frac{\left(1-\chi_i^a\right)}{p} \right)$$

H_{21} , H_{31} and H_{32} are obtained from $H_{ji} = H_{ij}$.

V (3×3)

As before, V is obtained by inverting H , i.e., $V = H^{-1}$.

Printing out estimates for Model D2

After convergence, the elements of $\underset{\sim}{\theta}$ are

$$\theta_1 = \hat{\varphi}^a \quad , \qquad \theta_2 = \hat{p} \quad , \qquad \theta_3 = \hat{\varphi}^y \quad .$$

V is the estimated variance-covariance matrix for $\hat{\varphi}^a$, $\hat{p}$, and $\hat{\varphi}^y$.

Thus,

$$V_{11} = \text{var } \hat{\varphi}^a \quad , \quad V_{22} = \text{var } \hat{p} \quad , \quad V_{33} = \text{var } \hat{\varphi}^y \quad ,$$

$$V_{12} = \text{cov}(\hat{\varphi}^a, \hat{p}) \quad , \quad V_{13} = \text{cov}(\hat{\varphi}^a, \hat{\varphi}^y) \quad , \quad V_{23} = \text{cov}(\hat{p}, \hat{\varphi}^y) \quad .$$

These estimates, standard errors and covariances (correlations?) should be printed.

Estimates of M_i , N_i^a , B_i^a

$$\hat{M}_i = \frac{m_i + z_i}{1-\hat{q}\ x_i^a} \quad , \quad \hat{U}_i^a = \frac{u_i^a}{\hat{p}} \quad , \quad \hat{N}_i^a = \hat{U}_i^a + \hat{M}_i \quad , \quad i=2,\ldots,s \quad .$$

$$\hat{B}_i^a = \hat{U}_{i+1}^a - \hat{\varphi}^a\ \hat{q}\ \hat{U}_i^a + R_i^y\ \hat{\varphi}^y \quad , \qquad i=2,\ldots,s-1 \quad .$$

Corresponding variances and covariances are obtained using the matrix $\underset{3\times s-1}{W} = V\ X$, where X is defined below.

Matrix X (3 by s-1)

Row 1 of X

$$X_{1j} = q\,\chi^{a}_{j+1} \sum_{i=1}^{j} \frac{(q\,\varphi^{a})^{j-i}}{1-\chi^{a}_{i}} \left\{ \frac{\delta_i\, R^{a}_{i}}{\chi^{a}_{i}} + \varphi^{y}\, q\, \delta_{i+1} \frac{R^{y}_{i}}{\chi^{y}_{i}} + \underbrace{\frac{pq\, \delta_i\, M_i}{1-q\,\chi^{a}_{i}}}_{\text{term vanishes for } i=1} \right\}$$

$$j=1,\ldots,s-1$$

Row 2 of X

$$X_{2j} = -q\,\chi^{a}_{j+1} \sum_{i=1}^{j} \frac{(q\,\varphi^{a})^{j-i}}{1-q\,\chi^{a}_{i+1}} \left[\frac{1}{q} \left\{ \delta_i - \frac{(1-\chi^{a}_{i})}{p} \right\} \left\{ \frac{R^{a}_{i}}{\chi^{a}_{i}} + \frac{\varphi^{y}}{\varphi^{a}} \frac{R^{y}_{i}}{\chi^{y}_{i}} \right\} + \underbrace{\frac{\delta_i\, M_i\, p}{1-q\,\chi^{a}_{i}}}_{\text{term vanishes for } i=1} \right]$$

$$j=1,\ldots,s-1$$

Row 3 of X

$$X_{3j} = q\,\chi^{a}_{j+1} \sum_{i=1}^{j} \left(q\,\varphi^{a}\right)^{j-i} \frac{R^{y}_{i}}{\chi^{y}_{i}} \qquad j=1,\ldots,s-1 \;.$$

Matrix W (3 by s-1)

Compute W = VX by matrix multiplication.

Also, compute $P_i = \delta_i - \dfrac{(1-q\,\chi^{a}_{i})}{p}$, $\qquad i=2,\ldots,s$

(recall $\delta_s=0$)

Then,

$$\text{var}\left(\hat{M}_i\right) = \frac{M_i}{1-q\ \chi_i^a}\left\{q\ \chi_i^a + \left(\frac{q\ \delta_i}{\varphi^a}\right)^2 \frac{M_i\ V_{11}}{1-q\ \chi_i^a} + P_i^2\ \frac{M_i\ V_{22}}{1-q\ \chi_i^a}\right.$$

$$\left. - 2\left(\frac{q\ \delta_i}{\varphi^a}\right) W_{1,i-1} + 2\ P_i\ W_{2,i-1} - 2\ P_i\left(\frac{q\ \delta_i}{\varphi^a}\right)\frac{M_i\ V_{12}}{1-q\ \chi_i^a}\right\}$$

$$i=2,\ldots,s$$

$$\text{Cov}\left(\hat{M}_i,\hat{M}_j\right) = \frac{q\ \chi_j^a\ M_i}{1-q\ \chi_i^a}\left(q\ \varphi^a\right)^{j-i} - \frac{M_j}{1-q\ \chi_j^a}\left[\frac{q\ \delta_j}{\varphi^a}\ W_{1,i-1} - P_j\ W_{2,i-1}\right]$$

$$- \frac{q\ \delta_i\ M_i}{\varphi^a\left(1-q\ \chi_i^a\right)}\left[W_{1,j-1} - \frac{q\ \delta_j\ M_j}{\varphi^a\left(1-q\ \chi_j^a\right)}\ V_{11} + \frac{P_j\ M_j\ V_{12}}{1-q\ \chi_j^a}\right]$$

$$+ \frac{P_i\ M_i}{1-q\ \chi_i^a}\left[W_{2,j-1} - \frac{q\ \delta_j\ M_j}{\varphi^a\left(1-q\ \chi_j^a\right)}\ V_{12} + \frac{P_j\ M_j\ V_{22}}{1-q\ \chi_j^a}\right]$$

$$\text{for } i<j,\ i=2,\ldots,s-1,\ j=i+1,\ldots,s$$

Note that in $\text{Cov}\left(\hat{M}_i,\hat{M}_s\right)$ all terms involving δ_s will be 0 .

$$\text{Cov}\left(\hat{p},M_i\right) = W_{2,i-1} - \frac{q\ \delta_i\ M_i}{1-\chi_{i-1}^a}\ V_{12} + \frac{P_i\ M_i\ V_{22}}{1-q\ \chi_i^a} \qquad i=2,\ldots,s$$

$$\text{Var}\left(\hat{N}_i^a\right) = \text{Var}\left(\hat{M}_i\right) + \frac{U_i^a}{p}\left\{q + \frac{U_i^a}{p}\ V_{22} - 2\ \text{Cov}\left(\hat{p},\hat{M}_i\right)\right\} \qquad i=2,\ldots,s$$

$$\text{Cov}\left(\hat{N}_i^a,\hat{N}_j^a\right) = \text{Cov}\left(\hat{M}_i,\hat{M}_j\right) - \frac{U_i^a}{p}\text{Cov}\left(\hat{p},\hat{M}_j\right) - \frac{U_j^a}{p}\text{Cov}\left(\hat{p},\hat{M}_i\right)$$

$$+ \frac{U_i^a U_j^a V_{22}}{p^2} \qquad \begin{array}{l} i=2,\ldots,s-1 \\ j=i+1,\ldots,s \end{array}$$

$$\text{Var}\left(\hat{B}_i^a\right) = \frac{q\,U_{i+1}^a}{p} + q\,\varphi^a\,U_i^a\left(1 + \frac{q\,\varphi^a}{p}\right) + \left(\frac{U_{i+1}^a - \varphi^a\,U_i^a}{p}\right)^2 V_{22}$$

$$+ \left(q\,U_i^a\right)^2 V_{11} + 2\left(\frac{U_{i+1}^a - \varphi^a\,U_i^a}{p}\right)\left(q\,U_i^a\right) V_{12} + \left(R_i^y\right)^2 V_{33}$$

$$+ 2R_i^y\left\{\left(\frac{\varphi^a\,U_i^a - U_{i+1}^a}{p}\right)V_{23} - U_{i-1}^a\,q\,V_{13}\right\} \qquad i=2,\ldots,s-1$$

$$\left(\begin{array}{c}\text{OMIT if} \\ s=3\end{array}\right) \quad \text{Cov}\left(\hat{B}_{i-1}^a,\hat{B}_i^a\right) = -\frac{q\,\varphi^a\,U_i^a}{p} + \left(\frac{U_i^a - \varphi^a\,U_{i-1}^a}{p}\right)\left(\frac{U_{i+1}^a - \varphi^a\,U_i^a}{p}\right)V_{22}$$

$$+ q^2\,U_{i-1}^a\,U_i^a\,V_{11}$$

$$+ \left\{q\,U_{i-1}^a\left(\frac{U_{i+1}^a - \varphi^a\,U_i^a}{p}\right) + q\,U_i^a\left(\frac{U_i^a - \varphi^a\,U_{i-1}^a}{p}\right)\right\}V_{12}$$

$$+ R_{i-1}^y\,R_i^y\,V_{33}$$

$$+ R_i^y\left\{\left(\frac{\varphi^a\,U_{i-1}^a - U_i^a}{p}\right)V_{23} - U_{i-1}^a\,q\,V_{13}\right\}$$

$$+ R^y_{i-1} \left\{ \left(\frac{\varphi^a U^a_i - U^a_{i+1}}{p} \right) V_{23} - U^a_i q V_{13} \right\}$$

$$i=3,\ldots,s-1$$

$$\begin{bmatrix} \text{OMIT if} \\ s=3 \text{ or } 4 \end{bmatrix} \quad \mathrm{Cov}\left(\hat{B}^a_i, \hat{B}^a_j\right) = \left(\frac{U^a_{i+1} - \varphi^a U^a_i}{p} \right) \left(\frac{U^a_{j+1} - \varphi^a U^a_j}{p} \right) V_{22}$$

$$+ q^2 U^a_i U^a_j V_{11}$$

$$+ \left[q U^a_i \left\{ \frac{U^a_{j+1} - \varphi^a U^a_j}{p} \right\} + q U^a_j \left\{ \frac{U^a_{i+1} - \varphi^a U^a_i}{p} \right\} V_{12} \right.$$

$$+ R^y_i R^y_j V_{33}$$

$$+ R^y_i \left\{ \left(\frac{\varphi^a U^a_j - U^a_{j+1}}{p} \right) V_{23} - U^a_j q V_{13} \right\}$$

$$+ R^y_j \left\{ \left(\frac{\varphi^a U^a_i - U^a_{i+1}}{p} \right) V_{23} - U^a_i q V_{13} \right\}$$

$$i=2,\ldots,s-3, \quad j=i+2,\ldots,s-1$$

Note that formulae for $\hat{B}^a_i$, variances and covariances have been included for Models B2 and D2, but are not given in Pollock (1981). Pollock (1981) states that B^a_i refers to recruitment of adults through <u>immigration</u> <u>only</u> [page 523 (first paragraph)]. However, estimators $\hat{B}_i^a$ defined on page 20 here, seem to include recruitment through survival of unmarked young in year i . If the $\hat{B}_i^a$ on page 20 do not seem to be meaningful quantities, then the program need not compute the $\hat{B}_i^a$, variances and covariances.

Testing between Models D2, B2 and A2.

(i) **Test of Model B2 versus Model A2**

Let $\hat{p}_{i,B2}$, $\hat{q}_{i,B2}$, $\hat{\chi}_{i,B2}^a$ and $\hat{\chi}_{i,B2}^y$ be the model B2 estimates of p_i , q_i , χ_i^a and χ_i^y , respectively.

Let $$\hat{\rho}_{i,B2} = \frac{\hat{p}_{i,B2}}{1-\hat{q}_{i,B2}\,\hat{\chi}_{i,B2}^a} \quad , \quad i=2,\dots,s-1 \quad .$$

Compute $$T_{1i}^a = \frac{[r_i^a - R_i^a(1-\hat{\chi}_{i,B2}^a)]^2}{R_i^a\hat{\chi}_{i,B2}^a(1-\hat{\chi}_{i,B2}^a)} \quad , \quad i=1,\dots,s-1$$

$$T_{2i} = \frac{[m_i-(m_i+z_i)\hat{\rho}_{i,B2}]^2}{(m_i+z_i)\hat{\rho}_{i,B2}(1-\hat{\rho}_{i,B2})} \quad , \quad i=2,\dots,s-1$$

$$T_{1i}^y = \frac{[r_i^y - R_i^y(1-\hat{\chi}_{i,B2}^y)]^2}{R_i^y\,\hat{\chi}_{i,B2}^y(1-\hat{\chi}_{i,B2}^y)} \quad , \quad i=1,\dots,s-1$$

Print out individual chi-square values $T_{11}^a,\dots,T_{1,s-1}^a$,

$T_{22},\dots,T_{2,s-1}$,

and $T_{11}^y,\dots,T_{1,s-1}^y$.

Also print out

$$T_1^a = \sum_{i=1}^{s-1} T_{1i}^a \quad , \quad T_2 = \sum_{i=2}^{s-1} T_{2i} \quad , \quad T_1^y = \sum_{i=1}^{s-1} T_{1i}^y \quad ,$$

and "Total chi-square" $= T_1^a + T_2 + T_1^y$

"Degrees of freedom" $= 2s - 5$

"Probability" = (computed in usual way)

(ii) <u>**Model D2 versus Model A2**</u>.

Let $\hat{p}_{D2}$, $\hat{q}_{D2}$, $\hat{x}^a_{i,D2}$ and $\hat{x}^y_{i,D2}$ be the model D2 estimates of p , q , x^a_i and x^y_i , respectively.

Let
$$\hat{\rho}_{i,D2} = \frac{\hat{p}_{D2}}{1-\hat{q}_{D2}\,\hat{x}^a_{i,D2}} \quad , \quad i=2,\ldots,s-1 \quad .$$

Compute
$$T^a_{1i} = \frac{\left[r^a_i - R^a_i\left(1-\hat{x}^a_{i,D2}\right)\right]^2}{R^a_i\hat{x}^a_{i,D2}\left(1-\hat{x}^a_{i,D2}\right)} \quad , \quad i=1,\ldots,s-1$$

$$T_{2i} = \frac{\left[m_i - \left(m_i+z_i\right)\hat{\rho}_{i,D2}\right]^2}{\left(m_i+z_i\right)\hat{\rho}_{i,D2}\left(1-\hat{\rho}_{i,D2}\right)} \quad , \quad i=2,\ldots,s-1$$

$$T^y_{1i} = \frac{\left[r^y_i - R^y_i\left(1-\hat{x}^y_{i,D2}\right)\right]^2}{R^y_i\,\hat{x}^y_{i,D2}\left(1-\hat{x}^y_{i,D2}\right)} \quad , \quad i=1,\ldots,s-1$$

Print out
$$T^a_{11},\ldots,T^a_{1,s-1} \quad ,$$
$$T_{22},\ldots,T_{2,s-1} \quad ,$$
$$T^y_{11},\ldots,T^y_{1,s-1} \quad ,$$

and $T^a_1 = \sum_1^{s-1} T^a_{1i}$, $T_2 = \sum_2^{s-1} T_{2i}$, and $T^y_1 = \sum_1^{s-1} T^y_{1i}$.

Also print out

"Total chi-square" $= T_1^a + T_2 + T_1^y$

"Degrees of freedom" $= 3s - 7$

"Probability" $= \ldots$

(iii) <u>**Model D2 versus Model B2**</u>

Compute
$$L_{1i}^a = -2\left\{r_i^a \log_e\left(\frac{1-\hat{\chi}_{i,D2}^a}{1-\hat{\chi}_{i,B2}^a}\right) + \left(R_i^a - r_i^a\right) \log_e \frac{\hat{\chi}_{i,D2}^a}{\hat{\chi}_{i,B2}^a}\right\}$$

$$i=1,\ldots,s-1$$

$$L_{2i} = -2\left\{m_i \log_e \frac{\hat{\rho}_{i,D2}}{\hat{\rho}_{i,B2}} + z_i \log_e\left(\frac{1-\hat{\rho}_{i,D2}}{1-\hat{\rho}_{i,B2}}\right)\right\}$$

$$i=2,\ldots,s-1$$

$$L_{1i}^y = -2\left\{r_i^y \log_e\left(\frac{1-\hat{\chi}_{i,D2}^y}{1-\hat{\chi}_{i,B2}^y}\right) + \left(R_i^y - r_i^y\right) \log_e \frac{\hat{\chi}_{i,D2}^y}{\hat{\chi}_{i,B2}^y}\right\}$$

$$i=1,\ldots,s-1 \quad .$$

Print out $L_{11}^a,\ldots,L_{1,s-1}^a$

$L_{22},\ldots,L_{2,s-1}$

$L_{11}^y,\ldots,L_{1,s-1}^y$

and $L_1^a = \sum_1^{s-1} L_{1i}^a \quad , \quad L_2 = \sum_2^{s-1} L_{2i} \quad , \quad L_1^y = \sum_1^{s-1} L_{1i}^y \quad .$

Also print

"Total chi-square =" $L_1^a + L_2 + L_1^y$

"Degrees of freedom =" s-2

"Probability =" .

Checking for Small Expectations.

In carrying out these tests, before computing individual T_{ij} or L_{ij} values, it will be necessary to check for small expectations as follows:

(i) B2 versus A2

Check for values <2 :

$$R_i^a \hat{\chi}_{i,B2}^a \ , \quad R_i^a(1-\hat{\chi}_{i,B2}^a) \ , \quad R_i^a - r_i^a \ , \ r_i^a \ , \qquad i=1,\ldots,s-1$$

$$R_i^y \hat{\chi}_{i,B2}^y \ , \quad R_i^y(1-\hat{\chi}_{i,B2}^y) \ , \quad R_i^y - r_i^y \ , \ r_i^y \ , \qquad i=1,\ldots,s-1$$

$$(m_i+z_i)\hat{\rho}_{i,B2} \ , \quad (m_i+z_i)(1-\hat{\rho}_{i,B2}) \ , \quad m_i, z_i \ , \qquad i=2,\ldots,s-1$$

(ii) D2 versus A2

As for B2 versus A2 above, but replacing B2 estimates in formulae with D2 estimates (i.e., replacing $\hat{\chi}_{i,B2}^a$ with $\hat{\chi}_{i,D2}^a$, etc.).

(iii) D2 versus B2

Check for values <2 :

$$R_i^a \hat{\chi}_{i,D2}^a \ , \quad R_i^a(1-\hat{\chi}_{i,D2}^a) \ , \quad R_i^a \hat{\chi}_{i,B2}^a \ , \quad R_i^a(1-\hat{\chi}_{i,B2}^a) \ ,$$

$$i=1,\ldots,s-1$$

$$R_i^y \hat{\chi}_{i,D2}^y \ , \quad R_i^y(1-\hat{\chi}_{i,D2}^y) \ , \quad R_i^y \hat{\chi}_{i,B2}^y \ , \quad R_i^y(1-\hat{\chi}_{i,B2}^y) \ ,$$

$$i=1,\ldots,s-1$$

$$(m_i+z_i)\hat{\rho}_{i,D2} \quad , \quad (m_i+z_i)(1-\hat{\rho}_{i,D2}) \ ,$$

$$(m_i+z_i)\hat{\rho}_{i,B2} \quad , \quad (m_i+z_i)(1-\hat{\rho}_{i,B2})$$

$$i=2,\ldots,s-1$$

When expectations <2 are found, it will be necessary to pool before computing the T_{ij} or L_{ij} value, for example, as described for testing between models D, B and A. Alternatively, the component T_{ij} or L_{ij} could be omitted entirely (and a degree of freedom subtracted from the degrees of freedom for the total chi-square).

Goodness of fit tests

(i) **Test of fit to Model B2**

"Test of fit" chi-square = chi-square for B2 versus A2 + chi-square for test of fit to A2.

df for test of fit to B = df for B2 vs A2 + df for test of fit to A2.

(ii) **Test of fit to Model D2**

"Test of fit" chi-square = chi-square for D2 vs A2 + chi-square for test of fit to A2.

df for test of fit to D2 = df for D2 vs A2 + df for test of fit to A2.

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