ON POSITIVE AUTOREGRESSIVE SEQUENCES

by

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Key Words: Autoregressive processes, Laplace Transforms of distributions.

This research is supported by Office of Naval Research Grant No. N00014-83-K-0352.

SUMMARY

A stationary sequence of non-negative random variables $\{X_n\}$, with marginal df F, is to be generated by the familiar autorgressive model $X_n = \sum_1^k a_1 X_{n-j} + \epsilon_n$, where the $\{\epsilon_n\}$ are iid with df G. Certain periodic cases are excluded from consideration. A class $\mathcal{N}_1$ of distributions which are reasonably "pleasant" from a computational viewpoint is defined, and it is shown that if both F and G belong to $\mathcal{N}_1$ then $k=1$; i.e. only the simple Markov case of the autoregressive model permits both F and G to be "pleasant". Other questions related to the generation of non-negative $\{X_n\}$ are also discussed.

¶1. INTRODUCTION

The main question considered in this paper is as follows. Let a stationary sequence $\{X_n\}$ be generated by the familiar autoregressive model, of order $k \geq 1$,

$$X_n = a_1 X_{n-1} + a_{n-2} X_{n-2} + \cdots + a_k X_{n-k} + \epsilon_n \tag{1.1}$$

where the $\{\epsilon_n\}$ are assumed to be iid. Let F be the stationary marginal df of the $\{X_n\}$ and let G be the df of the $\{\epsilon_n\}$. It is required that the $\{X_n\}$ be almost surely non-negative and that F and G be, in some reasonable sense, "pleasant" distributions. For example, it would be "pleasant" if both F and G had rational Laplace-Stieltjes Transforms (L-S. T.) as defined in (2.1)

Later in this note we shall make explicit what, for present purposes, we shall regard as a "pleasant" distribution; our definition will include much more than merely those with rational transforms. We shall then show that, at least for a substantial class of autoregressive models, the desired objective can, unfortunately, only be attained when (1.1) is a Markov model, that is, when $k = 1$.

This investigation was triggered by a colloquium given in Chapel Hill by P. A. Lewis, in which he described extensive work which he and colleagues have been doing for the computer generation of "interesting" stationary sequences of dependent positive random variables. Their object was to obtain, with reasonably tractable algorithms, sequences $\{X_n\}$ which could be used in simulation studies of queueing, inventory, and other related models encountered in Operations Research. For such applicaitons it is very desirable that both F and G be tractable. In particular, a df with a ratiuonal L-S. T. is particularly attractive since it has a pdf expressible in terms of exponential and polynomial func5tions; such a pdf is convenient for computer use.

The feasibility of the autoregressive scheme (1.1) was considered by Lewis, but only for the Markov case: $k = 1$. Preliminary announcements of this work was given in Lawrence & Lewis (1979). However, the present author conjectured at that colloquium that it is only possible to do what is wanted if $k = 1$ (and, of course, the process is Markov). The present note is directed at the partial substantiation of that, perhaps too sweeping, conjecture.

In ¶2 we introduce the notion of a g-singularity and use this to define the class $\mathcal{N}_1$ of distributions which correspond to what we have loosely been calling "pleasant". Very roughly, they are such that $F^*(s)$, the L-S. T. of $F(x)$ has at least one, but at most a finite number, of singularities on the negative real axis. In ¶3 we examine general conditions which will need to be fulfilled if the $\{X_n\}$ are to be non-negative; not surprisingly we need the σ_m of (3.2) to be ≥ 0 for all m. We give a necessary condition on the coefficients $\{a_k\}$, involving an infinite sequence of determinants. In ¶3 we also introduce Assumption A about the autoregressive scheme (1.1); by making it we exclude from our discussion certain periodic models for which our present methods are inadequate. In ¶4 we engage in some function theory and discover necessary conditions on G (whether or not Assumption A holds true) if $F \in \mathcal{N}_1$ and the $\{X_n\}$ are non-negative. We show that G^* can have no limit points of g-singularities on the negative real axis, we introduce a "power-function" which measures the "amount of singularity" of G^* in an interval, and we obtain an order of magnitude of this "power-function". Finally, in ¶5, we prove the following:

THEOREM *If* Assumption A *is valid, if both* F *and* G *belong to* $\mathcal{N}_1$, *and if the* $\{X_n\}$ *are a.s. non-negative, then we must have* k = 1, *i.e. the autoregressive scheme generates a Markov process.*

¶2. CLASSES OF "PLEASANT" DISTRIBUTIONS

For complex s with $\Re s \geq 0$ we denote the L-S. T. of a df $F(x)$ thus:

$$F^*(s) = \int_{0-}^{\infty} e^{-sx} F(dx). \tag{2.1}$$

It is well-known (cf Widder, 1946, p. 58) that $F^*(s)$ will be analytic in some half-plane $\Re s > -c$, say, where $c \geq 0$ is real and where $F^*(s)$ will necessarily have a singularity at $-c$. For real s, $F^*(s)$ is bounded and decreasing in any interval $(-c+\delta, \infty)$, for $\delta > 0$.

It was originally intended to take as our class of "pleasant" distributions exactly those F for which $F^*(s)$ is rational. However, such a class excludes many distributions which would be found quite acceptable for computer use. For example, the case

$$F^*(s) = \frac{1}{\sqrt{1+s}} \tag{2.2}$$

which corresponds to a pdf $e^{-x}/\sqrt{(2\pi x)}$, $x>0$, should by any reasonable argument, be included in a class branded "pleasant".

At this stage we must introduce some unconventional terminology to describe singularities of a function $f(s)$, say, of the complex variable s. Suppose $f(s)$ to have an isolated singularity or zero at ς and suppose there to be a finite $\mu>0$ such that, as $s \to \varsigma$, $|s-\varsigma|^{\mu}|f(s)|$ tends to a finite limit <u>different from zero</u>. In such a case we shall say $f(s)$ has a <u>g-singularity</u> at ς, and we shall call μ the <u>power</u> of $f(s)$ at ς. If the power is positive we may say $f(s)$ has a g-pole at ς; if the power is negative we say $f(s)$ has a g-zero. In particular, for example, an ordinary zero of $f(s)$ at ς would yield a power $= -1$; a pole of order k (an integer ≥ 1) at ς would yield a power $= k$. However, as (2.2) above shows, non-integral powers can arise: the example (2.2) involves a g-pole at $\varsigma = -1$, of power $\frac{1}{2}$.

We shall occasionally write $\mathcal{P}[\varsigma, f(s)]$ for the power of $f(s)$ at ς; thus. if ς be a regular point of $f(s)$, we have $\mathcal{P}[\varsigma, f(s)] = 0$.

It is possible to generalize the preceding ideas considerably. We could, for example, seek a vector (μ_1, μ_2, μ_3) such that

$$|s-\varsigma|^{\mu_1} |\log|s-\varsigma||^{\mu_2} \; |\log\log|s-\varsigma||^{\mu_3} \; |f(s)| \tag{2.3}$$

tends to a non-zero limit as $s \to \varsigma$. By introducing a dummy variable $\lambda > 0$ we could then define the <u>power</u> of $f(s)$ as the polynomial $\mu_1 + \mu_2\lambda^2 + \mu_3\lambda^3 = \mu(\lambda)$, say. We could say ς is a g-pole if $\mu(\lambda)$ us strictly positive for all small λ, a g-zero if $\mu(\lambda)$ is strictly negative for all small λ. We believe the theory of the present note could be developed to cover this more general concept of g-singularity and power, but, in the interest of simplicity, we shall proceed on the earlier basis of a real-valued power. However, it might be mentioned that other scales of growth could be used in (2.3) besides the logarithmic one discussed.

For any small angle $\alpha > 0$, define $\mathcal{W}(\alpha) = \{s | \pi-\alpha \leq \text{Arg}\, s \leq \pi+\alpha\}$, <u>including 0</u> in this thin sector. We are now able to define the classes of distributions which are important for the present work.

<u>Definition 1</u> A df F belongs to the class $\mathcal{N}_0$ if, for some small $\alpha > 0$, all zeros and singularities of $F^*(s)$ within the sector $\mathcal{W}(\alpha)$ lie on the negative real axis.

<u>Definition 2</u> A df F, in the class $\mathcal{N}_0$, belongs to the class $\mathcal{N}_1$ if it has at least one, but at most a finite number, of g-singularities in $\mathcal{W}(\alpha)$, for some $\alpha > 0$.

The class $\mathcal{N}_1$ constitutes our class of "pleasant" distributions.

¶3 POSITIVE VARIABLES AND AUTOREGRESSIVE SEQUENCES

We shall be considering the autoregressive scheme (1.1) with real coefficients $a_1, a_2, \ldots, a_k$. It will be assumed that (1.1) will generate a stationary sequence $\{X_n\}$. This, as is well-known, requires that the equation

$$z^k - a_1 z^{k-1} - a_2 z^{k-2} - \ldots - a_k = 0 \tag{3.1}$$

have all its k roots strictly within the unit circle.

It is then possible to represent the $\{X_n\}$ in the following way

$$X_n = \epsilon_n + \sigma_1 \epsilon_{n-1} + \sigma_2 \epsilon_{n-2} + \cdots \tag{3.2}$$

where the numbers $\{\sigma_n\}$ are calculable in terms of the roots of (3.1). Indeed, when the roots are all distinct, $\lambda_1, \lambda_2, \ldots \lambda_k$, say, one has the straightforward formula:

$$X_n = A_1 \lambda_1^n + A_2 \lambda_2^n + \cdots + A_k^n \tag{3.3}$$

where the constants $A_1, A_2, \ldots, A_k$ are easy to obtain. Unfortunately, these roots may not be distinct, some may be complex, and then a more elaborate formula than (3.3) becomes necessary (Box and Jenkins (1970) have much general information on autoregressive models). In general there will be real numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_p$, say, and functions $\Psi_1(m), \Psi_2(m), \ldots, \Psi_p(m)$ such that

$$\sigma_m = \sum_{j=1}^{p} \lambda_j^m \; \Psi_j(m) \; . \tag{3.4}$$

The functions $\Psi_j(m)$ have the form

$$\Psi_j(m) = \sum_{t=1}^{t_j} A_{jt}(m)\Theta(w_{jt}m) \tag{3.5}$$

where: (i) each $A_{jt}(m)$ is a polynomial in m, of degree d_{jt}, say; (ii) $\Theta(\cdot)$ stands for either $\sin(\cdot)$ or $\cos(\cdot)$; (iii) the w_{jt} are periods, related to the arguments of complex roots of (3.1). Because λ_1 exceeds the remaining $\{\lambda_j\}$, it is evident from (3.4) that, as $m \to \infty$,

$$\sigma_m \sim \lambda_1^m \Psi_1(m)$$

However, let $d_1 = \max\{d_{11}, d_{12}, \ldots, d_{1t}\}$. Then evidently, as $m \to \infty$,

$$\Psi_1(m) \sim m^{d_1}\Phi_1(m),$$

where $\Phi_1(m)$ has the form, for some constants $\{C_{1j}\}$,

$$\Phi_1(m) = C_{11} + C_{12}\Theta(w_{11}m) + \text{similar terms.} \tag{3.6}$$

This function $\Phi_1(m)$ in important to the asymptotic behavior of σ_m; we shall see below that, if the $\{X_n\}$ are to be non-negative, we must have $\sigma_m \geq 0$ for all m. Thus we may never have $\Phi_1(m) < 0$.

The trigonometric terms in (3.6) only arise when there are roots of (3.1) at points on the circle $|z| = \lambda_1$ (>0) which are not at the real point λ_1. When such roots exist the autoregressive process exhibits some curious periodic (or almost periodic) aspects, and needs far more elaborate treatment that we can manage in the present note. Thus, so far as Theorem 1 is concerned, we must introduce the following:

<u>Assumption A</u> The roots of (3.1) of maximum modulus are all located at the positive real point λ_1. Consequently $\Phi_1(m) = C_{11}$, a constant.

Subject to Assumption A we may then claim that, as $m \to \infty$,

$$\sigma_m \sim \lambda_1^m m^{d_1} C_{11}. \tag{3.7}$$

Even if Assumption A does not hold, it is clear that we can always find a λ, $0 < \lambda < 1$, such that $\sigma_m = O(\lambda^m)$.

__Lemma__ __3.1__ *If* $\mathcal{E} \log^+|\epsilon_m| < \infty$ *then the series* (3.2) *is almost surely absolutely convergent, and, under* Assumption A, *conversely.*

__Proof__ For any $h > 0$, if $\mathcal{E} \log^+|\epsilon_m| < \infty$, then $\sum_0^\infty P\{\log^+|\epsilon_n| \geq nh\} < \infty$; this implies that $\sum_0^\infty P\{|\epsilon_n| \geq e^{nh}\} < \infty$. Thus, by the Borel-Cantelli lemma, almost surely, $|\epsilon_n| \geq e^{nh}$ for only finitely many values of n. Since $\sigma_m = O(\lambda^m)$ for some $0 < \lambda < 1$, it is clear that (3.2) will converge as claimed because, with only a finite number of exceptions, $|\sigma_m \epsilon_{n-m}| \leq |\sigma_m| e^{mh}$.

Conversely, if (3.2) is almost surely convergent, $|\sigma_m \epsilon_{n-m}| > 1$ for only a finite number of values of n. Hence, by the converse to the Borel-Cantelli Lemma,

$$\sum_0^\infty P\{|\epsilon_{n-m}| > |\sigma_m|^{-1}\} < \infty.$$

If Assumption A holds, from (3.7) we can infer the existence of $\eta > 0$ such that $|\sigma_m|^{-1} < e^{\eta m}$ for all large m. Hence

$$\sum_0^\infty P\{|\epsilon_{n-m}| > e^{\eta m}\} < \infty,$$

from which it is an easy deduction that $\mathcal{E} \log^+|\epsilon_m| < \infty$.

Because of this lemma we shall henceforth assume without comment that $\mathcal{E} \log^+|\epsilon_m| < \infty$ and that (3.2) is almost surely absolutely convergent.

Lemma 3.2 *Let* (1.1) *produce a stationary sequence* $\{X_n\}$ *with df* $F(x)$; *let* $F^*(s)$ *have at least one singularity in the finite complex plane. Then, if the* $\{X_n\}$ *are almost surely non-negative, it is necessary that* $\sigma_m \geq 0$ *for all* $n \geq 0$.

Proof If the non-negative $\{X_n\}$ were a.s. bounded above then $F^*(s)$ would be entire. But $F^*(s)$ is not entire, so the $\{X_n\}$ can be arbitrarily large.

Suppose that, with positive probability , $\sigma_n \epsilon_{-n}$ can be arbitrarily large and negative. Evidently

$$X_0 - \sigma_n \epsilon_{-n} = \sum_{\substack{j=0 \\ j \neq n}}^{\infty} \sigma_j \epsilon_{-j} \tag{3.8}$$

is a proper random variable, independent of ϵ_{-n}. Thus there is a large $\Delta > 0$ such that

$$\mathbf{P}\{X_0 - \sigma_n \epsilon_{-n} < \Delta\} > 0.$$

But we may suppose

$$\mathbf{P}\{\sigma_n \epsilon_{-n} < -2\Delta\} > 0,$$

and hence, by independence, $\mathbf{P}\{X_0 < -\Delta\} > 0$. This contradiction shows $\sigma_n \epsilon_{-n}$ must a.s. be bounded below, for every n. In particular, taking $n = 0$ we use $\sigma_0 = 1$ to infer that ϵ_0 (and hence every ϵ_n) is a.s. bounded below.

If there were any n for which $\sigma_n < 0$ it would thus be necessary that ϵ_{-n} be a.s. bounded above (since $\sigma_n \epsilon_{-n}$ is bounded below). This would imply that every ϵ_{-n} is bounded above and thence that X_0 is a.s. bounded above. We have seen this cannot be so. Thus $\sigma_n \geq 0$ for every n. Thus the lemma is proved. □

The requirement $\sigma_m \geq 0$, for all n, imposes restraints upon the $\{a_n\}$.

Indeed, it is possible to show (we omit the proof) that the $\{a_n\}$ must be such that every member of the following sequence of determinants must be non-negative:

$$|a_1|, \quad \begin{bmatrix} a_1 & a_2 \\ -1 & a_1 \end{bmatrix}, \quad \begin{bmatrix} a_1 & a_2 & a_3 \\ -1 & a_1 & a_2 \\ 0 & -1 & a_1 \end{bmatrix}, \quad \ldots, \text{ and so on}$$

in this sequence one takes $a_j = 0$ for all $j > k$, of course.

Lemma 3.3 *Under the conditions of Lemma 3.2 the $\{\epsilon_n\}$ must be a.s. non-negative.*

Proof Let $G(x)$ be the df of the iid $\{\epsilon_n\}$; let γ be the esssential infemum of the $\{\epsilon_n\}$. We have seen that $\gamma > -\infty$. Then choose $\delta > 0$ and $N(\delta)$ so that

$$P\left\{\left|\sum_{N(\delta)+1}^{\infty} \sigma_n \epsilon_{-n}\right| < \delta\right\} > 0.$$

This must be possible because (3.2) is a.s. convergent. But. on the other hand, $P\{\epsilon_{-n} < \gamma+\delta\} > 0$ for all n. Thus

$$P\left\{\sum_0^{\infty} \sigma_n \epsilon_{-n} < (\gamma+\delta)\left[\sum_0^{\infty} \sigma_n\right] + \delta\right\} > 0.$$

This implies. letting $\delta \downarrow 0$, that the essential infemum of X_0 is $\le \gamma \sum_0^\infty \sigma_n$. But the $\{X_n\}$ are assumed to be a.s. non-negative. Thus $\gamma \ge 0$ and the lemma is proved.□

Lemma 3.4 *Let $G(x)$ be the df of the non-negative $\{\epsilon_n\}$. Let $F^*(s)$ be analytic in $\Re s \ge -\varsigma$ where ς is a real singularity of $F^*(s)$. Then $G^*(s)$ is analytic in, at least, $\Re s > -\varsigma$.*

Proof For any $\gamma < \varsigma$ it must be true that $1 - F(x) = O(e^{-\gamma x})$, as $x \to \infty$. But (3.2) shows $\epsilon_0 \le X_0$, from which it is immediate that $1 - G(x) = O(e^{-\gamma x})$ also. Thus $G^*(s)$ is finite for all real $s > -\varsigma$ and the lemma follows.□

In the sequel $-\varsigma$ will always refer to the real singularity of $F^*(s)$ which is nearest to 0. We shall always assume that such a finite ς exists; for instance, it always does when $F^*(s)$ is rational. We shall assume $\varsigma > 0$, and this implies an exponential rate of decay of $1 - F(x)$ as $x \to \infty$.

¶4. SOME FUNCTION THEORY

Let $\{\tau_n\}$ be a rearrangement of the $\{\sigma_n\}$ into no-increasing order. Thus $\tau_0 \geq 1$. It is apparent, in view of the absolute convergence of (3.2), that if we set

$$Y_0 = \tau_0\epsilon_0 + \tau_1\epsilon_1 + \cdots \text{ ad inf} \tag{4.1}$$

then Y_0 will be distributed like X_0 with df $F(x)$. From (4.1), for all s in the half-plane $\Re s > -\varsigma$, we obtain the crucial relation

$$F^*(s) = \prod_{j=0}^{\infty} G^*(\tau_j s). \tag{4.2}$$

Lemma 4.1 *If* $F \in \mathcal{N}_0$ *and* $\varsigma > 0$ *then* $G \in \mathcal{N}_0$.

Proof With no loss of generality we may suppose $\varsigma = 1$ (a suitable scale change for the $\{X_n\}$ and the $\{\epsilon_n\}$ will accomplish this). Let $\mathcal{I}$ be the segment $(-\infty, -1]$ in the complex plane. Let $\alpha > 0$ be such that $F^*(s)$ is analytic and zero-free at every point of $\mathcal{W}(\alpha) - \mathcal{I}$.

Choose β, $0 < \beta < \alpha$. Then $F^*(s)$ is analytic in $\mathcal{W}(\alpha) - \mathcal{W}(\beta)$. Let us also, for ease, write $\mathcal{G}(\alpha,\beta,r)$ for the set of all complex s in $\mathcal{W}(\alpha) - \mathcal{W}(\beta)$ such that $|s| < r$. Lemma 3.4 shows that $G^*(s)$ will be analytic in $\Re s > -1$. $G^*(s)$ cannot vanish in $(-1,0]$. Thus, if α be small enough, $G^*(s)$ must be analytic and zero-free in $\mathcal{G}(\alpha,\beta,1)$.

It is possible that, for some integer $p \geq 1$, we have

$$\tau_0 = \tau_1 = \cdots = \tau_{p-1} > \tau_p \geq \tau_{p+1} \geq \cdots$$

Because of this possibility we rewrite (4.2) as

$$F^*(s) = \{G^*(\tau_0 s)\}^p \prod_{j=p}^{\infty} G^*(\tau_j s). \tag{4.3}$$

We then define

$$\rho = \frac{\tau_p}{\tau_0} = \sup_{j \geq p} \frac{\tau_j}{\tau_0} < 1. \tag{4.4}$$

Evidently all $G^*(\tau_j s)$ for $j \geq p$ are analytic and zero-free in $\mathcal{G}(\alpha,\beta,1/\rho\tau_0)$. It is then routine to show that

$$\prod_{j=p}^{\infty} G^*(\tau_j s) = \prod\nolimits_p(s), \text{ say,}$$

is also analytic and zero-free in $\mathcal{G}(\alpha,\beta,1/\rho\tau_0)$. But $F^*(s)$ is analytic and zero-free throughout $\mathcal{W}(\alpha) - \mathcal{W}(\beta)$. Thus we see from (4.3) that $G^*(\tau_0 s)$ must be analytic and zero-free in $\mathcal{G}(\alpha,\beta,1/\rho\tau_0)$. This implies $G^*(s)$ is analytic and zero-free in $\mathcal{G}(\alpha,\beta,1/\rho)$ and hence, for all $j \geq p$, that $G^*(\tau_j s)$ is analytic and zero-free in $\mathcal{G}(\alpha,\beta,1/\rho^2\tau_0)$.

The argument can plainly recycle. We shall have $\Pi_p(s)$ analytic and zero-free in $\mathcal{G}(\alpha,\beta,1/\rho^2\tau_0)$ and deduce that $G^*(s)$ is analytic and zero-free in $\mathcal{G}(\alpha,\beta,1/\rho^2)$. It should be plain how this argument continues. Since β may be arbitrarily small, the lemma follows.□

A comment or two is in order. It must be understood that the conclusion $G \in \mathcal{N}_0$ is, of course, a consequence of the over-riding assumption that the autoregressive scheme is generating non-negative $\{X_n\}$. It might also be pointed out that the lemma could be proved in a very similar way if

$\varsigma = \infty$, which would mean that $F^*(s)$ is entire. However, we have no need for such a result in the sequel.

Lemma 4.1 *Let* $F \varepsilon \mathcal{N}_1$ *and* $\varsigma > 0$ *and suppose that the singularities which* $G^*(s)$ *may have on the negative real axis are g-singularities. Then these g-singularities of* $G^*(s)$ *must be, at most, countable; moreover they can have no finite limit points. The sum of the absolute powers of all the g-singularities of* $G^*(s)$ *in* $(-x,0)$, *for* $x > 0$, *is* $O(x^c)$ *for some constant* $c > 0$.

Proof It follows from Lemma 4.1 that $G \varepsilon \mathcal{N}_0$, thus we are supposing that, for small $\alpha > 0$, whatever singularities $G^*(s)$ may have in $\mathcal{W}(\alpha)$, they are on the negative real axis; they are g-singularities. Since $F \varepsilon \mathcal{N}_1$, it has a finite number of g-poles and g-zeros in $\mathcal{I} \equiv (-\infty, -1]$; we continue to let -1 be the location of the nearest singularity of $F^*(s)$ to 0. Let the g-poles of $F^*(s)$ in $\mathcal{I}$ be at

$$-\varsigma_1 > -\varsigma_2 > \cdots > -\varsigma_p$$

and let the g-zeros be at

$$-\xi_1 > -\xi_2 > \cdots > -\xi_p .$$

Let us also write

$$\mathcal{P}\{-\varsigma_j, F^*(s)\} = \pi_j > 0, \quad 1 \le j \le p$$

$$\mathcal{P}\{-\xi_j, F^*(s)\} = -\omega_j < 0, \; 1 \le j \le q .$$

It is entirely possible that $F^*(s)$ have no g-zeros in $\mathcal{I}$; in that case that argument which follows could be somewhat simplified.

Let $\mathcal{I}(r) \equiv \{s|\ |s| < r\} \bigcap \mathcal{W}(\alpha)$. Because $\tau_n \downarrow 0$, given any $R > 0$ we can find $N(R)$ such that

$$\prod_{N(R)}^{\infty} G^*(\tau_j s)$$

is analytic and zero-free in $\mathcal{I}(R)$. This is because $\tau_j|s| < 1$ for all large j and $G^*(s)$ is analytic and zero-free in $(-1, 0]$; the angle α can always be chosen small enough to ensure the validity of these claims. Thus, if it is known that $\Pi_p(s)$ has a g-singularity in $\mathcal{I}(R)$, then this g-singularity must come from the product

$$\prod_{j=p}^{N(R)-1} G^*(\tau_j s). \tag{4.5}$$

In particular, if $\Pi_p(s)$ has a g-singularity of power $\mu\,(\neq 0)$ at some $-\xi \varepsilon \mathcal{I}(R)$ then we must have, from (4.5), that as $s \rightarrow -\xi$

$$\mu \log|s+\xi| + \sum_{j=p}^{N(R)-1} \log|G^*(\tau_j s)|$$

tends to a finite limit.

If $-\tau_j\xi$ is not a g-singularity of $G^*(s)$ then $\log|G^*(\tau_j s)|$ tends to a finite limit as $s \rightarrow -\xi$. Let us therefore write $\mathfrak{J}$ for the set of suffices j such that $p \leq j \leq N(R) - 1$ and $-\tau_j\xi$ is a g-singularity of $G^*(s)$. Then

$$\mu \log|s+\xi| + \sum_{\mathfrak{J}} \log|G^*(\tau_j s)|$$

tends to a finite limit as $s \rightarrow -\xi$. Let $\mathcal{P}[-\xi,\ G^*(\tau_j s)] = \mu_j$, say. Then $\sum_{\mathfrak{J}} \mu_j = \mu$.

Now suppose that $G^*(s)$ has a g-pole at some real point $x_0 < 0$. Let us write $\mathcal{P}[-x_0,\ G^*(s)] = \mu_0$. Then $G^*(\tau_0 s)$ has a g-pole at $-x_0/\tau_0$ with the same power μ_0. Let us us also write $\mathfrak{P}[-x_0/\tau_0, F^*(s)] = \nu_0$. It is possible that $\nu_0 = 0$, in which case minor changes are necessary in the following argument. It is also possible that $\nu_0 = p\mu_0$. If we look at (4.3) we see that this implies that $\Pi_p(s)$ has zero power at $-x_0/\tau_0$: the g-pole of $F^*(s)$ is exactly "balanced" by that of $\{G^*(\tau_0 s)\}^p$ at $-x_0/\tau_0$.

However it is possible that $\nu_0 \neq p\mu_0$. In this case we are forced by (4.3) to conclude that $\Pi_p(s)$ has power $\nu_0 - p\mu_0$ at $-x_0/\tau_0$. Thus, by the argument we have already explained, there must be a set $\mathfrak{J}_0$, say, of suffices j such that

$$\sum_{\mathfrak{J}_0} \mathfrak{P}[-x_0/\tau_0, G^*(\tau_j s)] = \nu_0 - p\mu_0. \tag{4.6}$$

Moreover, if $j \notin \mathfrak{J}_0$, then $\mathfrak{P}[-x_0/\tau_0, G^*(\tau_j s)] = 0$. On the other hand, if $j \in \mathfrak{J}_0$ then $-\tau_j x_0/\tau_0$ *is a g*-singularity of $G^*(s)$. We can say that a g-singularity of $G^*(s)$ at $-x_0$ is "balanced" by one of $F^*(s)$ at $-x_0/\tau_0$, together with the finitely-many g-singularities of $G^*(s)$ at the locations

$$-\tau_j x_0/\tau_0 = -x_{0j}, \text{ say.}$$

Since $\tau_j \leq \rho\tau_0$, $x_{0j} < \rho x_0$ for all $j \in \mathfrak{J}_0$. Thus that part of the power of a g-singularity of $G^*(s)$ at $-x_0$ which is not immediately balanced by a g-singularity of $F^*(s)$ at $-x_0/\tau_0$ is balanced by a finite number of g-singularities of $G^*(s)$ which lie in the interval $[0, \rho x_0]$. Recall that $\rho < 1$ and that $G^*(s)$ is analytic and zero-free in $(-\tau_0, 0]$. Thus, if $\rho x_0 \leq \tau_0$, the only possibility is that the g-singularity of $G^*(s)$ at $-x_0$ is balanced exactly by that of $F^*(s)$ at $-x_0/\tau_0$. Suppose, on the other hand. that $\rho x_0 \geq \tau_0$; then there may well be numbers $\{-x_{0j}\}$ in $[-\rho x_0, -\tau_0]$ at which $G^*(s)$ has g-singularities.

However, it should be clear that each g-singularity of $G^*(s)$ at the locations $\{-x_{0j}\}$ is either wholly balanced by a g-singularity of $F^*(s)$ at $-x_{0j}/\tau_0$ or by further g-singularities of $G^*(s)$ at locations $\{-x_{1j}\}$, say, which fall in $[-\rho^2\lambda_0, -\tau_0]$, provided $\rho^2\lambda_0 \geq \tau_0$. If $\rho^2\lambda_0 < \tau_0$ then it must be that every $\{-x_{0j}/\tau_0\}$ is a singularity of $F^*(s)$ and there can be no "second generation" numbers $\{x_{1j}\}$.

Obviously, whatever the initial x_0 may be, this argument can be repeated a finite number of times until a "generation" of singularities of $G^*(s)$ is reached, each of which is balanced by a g-singularity of $F^*(s)$.

The first important consequence of this is that any such number x_0 must be of the form

$$x_0 = \left(\frac{\tau_0}{\tau_{j_1}}\right)\left(\frac{\tau_0}{\tau_{j_2}}\right) \cdots \left(\frac{\tau_0}{\tau_{j_k}}\right)\xi, \tag{4.7}$$

where $-\xi/\tau_0$ is a g-singularity of $F^*(s)$, k is a finite integer, and all the suffices $\{j_k\}$, amongst which there may be repetitions, satisfy $j_k \geq p$. Thus every factor $(\tau_0/\tau_j) > (1/\rho)$.

Since we are supposing that $F^*(s)$ has finitely many g-singularities, it follows that in any finite interval there can be only a finite number of numbers like (4.7). This shows that the g-singularities of $G^*(s)$ on the negative real axis must be enumerable and can have no finite limit points.

Next, for $k = 1, 2, \ldots$, let I_k be the interval $[1/\rho^{k-1}, 1/\rho^k)$. let

$$\mathfrak{P}_k = \sum_{x \varepsilon I_k} |\mathfrak{P}[-x, F^*(s)]|$$

and

$$\mathfrak{Q}_k = \sum_{x \varepsilon I_k} |\mathfrak{P}[-x, G^*(\tau_0 s)]|$$

The argument in which we have been engaged shows that for $k = 1, 2, \dots,$

$$\mathfrak{Q}_k \le \mathfrak{P}_k + \mathfrak{Q}_{k-1} + \mathfrak{Q}_{k-2} + \cdots + \mathfrak{Q}_1 \tag{4.8}$$

From this one can infer

$$\mathfrak{Q}_k \le \mathfrak{P}_k + \mathfrak{P}_{k-1} + 2\mathfrak{P}_{k-2} + 2^2\mathfrak{P}_{k-3} + 2^{k-2}\mathfrak{P}_1. \tag{4.9}$$

Since $\mathfrak{P}_k = 0$ for all large k, it follows that $\mathfrak{Q}_1 + \mathfrak{Q}_2 + \cdots + \mathfrak{Q}_k = O(2^k)$. Hence if, for all $x > 0$, we now write

$$\mathfrak{Q}(x) = \sum_{y \in (-x, 0]} |\mathfrak{P}[y, G^*(s)]|$$

it is possible to claim $\mathfrak{Q}(1/\rho^k) = O(2^k)$. From this it is an easy deduction that $\mathfrak{Q}(x) = O(x^c)$ where

$$c = \frac{\log 2}{\log(1/\rho)}.$$

This proves the lemma.□

¶5. PROOF OF THEOREM

Let $F \in \mathcal{N}_1$, and let $F^*(s)$ have g-poles at $-e^{\xi_1}, -e^{\xi_2}, \dots, -e^{\xi_p}$ on the negative real axis. We insist that $F^*(s)$ have at least one such g-pole. Moreover, as we have already explained, we can assume with no loss of generality that $\varsigma = e^{\xi_1}$ have any convenient value, so we shall arrange that $e^{\xi_1} = \tau_0$. Let $F^*(s)$ also have g-zeros at $-e^{\eta_1}, -e^{\eta_2}, \dots, -e^{\eta_q}$; there may be no such g-zeros, but if there is one at all, it must be true that $e^{\eta_1} > e^{\xi_1}$.

Write

$$\mathcal{F}(x) = \sum_{y\epsilon[-x,0]} \mathcal{P}[y, F^*(s)]$$

and

$$\mathcal{G}(x) = \sum_{y\epsilon[-x,0]} \mathcal{P}[y, G^*(s)] .$$

The discussion of ¶4 then shows that

$$\mathcal{F}(x) = \sum_{j=0}^{\infty} \mathcal{G}(\tau_j x), \tag{5.1}$$

for all x. But $\mathcal{F}(x) = 0$ for all $x < \tau_0$, from which we may deduce from (5.1) that $\mathcal{G}(x) = 0$ for all $x < 1$.

Let us now set $\tau_j = e^{-\kappa_j}$, so that $\kappa_j \uparrow \infty$, and also write

$$\Phi(t) \equiv \mathcal{F}(e^t), \ \Psi(t) \equiv \mathcal{G}(e^t).$$

Then $\Phi(t) = \Psi(t) = 0$ for all $t < 0$, and (5.1) gives

$$\Phi(t) = \sum_{j=0}^{\infty} \Psi(t-\kappa_j). \tag{5.2}$$

From Lemma 4.2, since $F \varepsilon \mathcal{N}_1$, we can infer that $\Psi(t) = O(e^{ct})$. Thus $\Psi(t)$ admits a Laplace-Stieltjes Transform for all s such that $\Re s > c$, and (5.2) yields that very useful result

$$\Phi^*(s) = \Big(\sum_{j=0}^{\infty} e^{-\kappa_j s}\Big) \Psi^*(s). \tag{5.3}$$

This result (5.3) has depended only on the hypothesis that $F \varepsilon \mathcal{N}_1$. However, let us at this point invoke the additional hypothesis of the theorem that $G \in \mathcal{N}_1$ also. Suppose that $G^*(s)$ has g-poles at $-e^{x_1} > -e^{x_2} > \ldots > -e^{x_m}$, and g-zeros at $-e^{y_1} > -e^{y_2} > \ldots > -e^{y_m}$.

Let us also write

$$\alpha_j = \mathcal{P}[-e^{\xi_j}, F^*(s)],$$

$$-\beta_j = \mathcal{P}[-e^{\eta_j}, F^*(s)],$$

$$-a_j = \mathcal{P}[-e^{x_j}, G^*(s)],$$

$$-b_j = \mathcal{P}[-e^{y_j}, G^*(s)],$$

Then (5.3) can be rewritten as:

$$\Big\{\sum_1^p \alpha_j e^{-s\xi_j} - \sum_1^q \beta_j e^{-s\eta_j}\Big\} = \Big\{\sum_0^\infty e^{-s\kappa_j}\Big\}\Big\{\sum_1^m a_j e^{-sx_j} - \sum_1^n b_j e^{-sy_j}\Big\} \tag{5.4}$$

which implies

$$\sum_1^p \alpha_j e^{-s\xi_j} + \Big\{\sum_0^\infty e^{-s\kappa_j}\Big\}\Big\{\sum_1^n b_j e^{-sy_j}\Big\}$$
$$= \sum_1^q \beta_j e^{-s\eta_j} + \Big\{\sum_0^\infty e^{-s\kappa_j}\Big\}\Big\{\sum_1^m a_j e^{-sx_j}\Big\} \tag{5.5}$$

Our argument will actually make no use of the powers α_j, β_j, a_j, b_j, etc., but depends <u>only</u> on the "locations" e^{ξ_j}, e^{η_j}, etc. Since there is at least one g-pole of $F^*(s)$ the term $\alpha_1 e^{-s\xi_1}$ on the left of (5.5) must be balanced by a sum of terms (or by a single term) on the right of (5.5). Since $\xi_j \neq \eta_j$ for every j, we deduce there must be at least one term in the sum $\sum_1^n a_j e^{-sx_j}$, *i.e.* $G^*(s)$ must have at least one g-pole. Let $a_1 e^{-sx_1}$ be such a term. Since $\kappa_j \to \infty$, a term like $a_1 e^{-s(x_1+\kappa_j)}$, which arises on the right of (5.5), can only be balanced, for κ_j large, by a term of the form

$$b_u e^{s(yu + xv)}.$$

Thus there must be at least one g-zero of $G^*(s)$. It follows that neither of the sets $(y_1, y_2, \ldots, y_n)$ nor $(x_1, x_2, \ldots, x_m)$ is empty. We shall now proceed as if $y_n < x_m$; it should be clear how the ensuing argument could be changed if $y_n > x_m$ (it is impossible to have $y_n = x_m$).

Then, for each value of j there must correspond integers u and k such that

$$x_m + \kappa_j = y_u + \kappa_k \tag{5.6}$$

Thus (5.6) provides a correspondence between the suffix j and the suffixes u and k. Let $\mathfrak{J}$ be <u>any</u> infinite set of positive integers. Let u^* be fixed, $1 \leq u^* \leq n$. Let $\mathfrak{J}(u^*)$ be the set of all $j \varepsilon \mathfrak{J}$ for which a k can be found so that

$$x_m + \kappa_j = y_{u*} + \kappa_k.$$

At least one of the sets $\mathfrak{J}(1)$, $\mathfrak{J}(2)$, ... , $\mathfrak{J}(n)$ must be infinite. Let $\mathfrak{J}(u)$ now be such an infinite set.

Under Assumption A we have for $1 > \lambda_1 > \lambda_2 > \cdots$ etc.,

$$\sigma_j = \lambda_1^j A_1(j) + \lambda_2^j A_2(j) + \ldots$$

where $A_1(j)$ is a polynomial of degree d_1 in j; $A_2(j)$ is a "polynomial" of degree d_2 with, however, some coefficents possibly involving cosine and sine terms, and similarly for $A_3(j)$, $A_4(j)$, and so on.

Let us set $\gamma = \log(1/\lambda_1)$. Then, for $j \in \mathfrak{J}(u)$ we have from (5.6) the relation

$$x_m - y_u = -j\gamma + \log\{A_1(j) + O([\lambda_2/\lambda_1]^j)\} \\ +k\gamma - \log\{A_1(k) + O([\lambda_2/\lambda_1]^k)\} \tag{5.7}$$

It is evident, since x_m and y_u are fixed, that $k \to \infty$ if $j \to \infty$ through $\mathfrak{J}(u)$. Furthermore,

$$j^{-1}\log\{A_1(j) + O([\lambda_2/\lambda_1]^j)\} \to 0, \text{ as } j \to \infty.$$

Thus we deduce from (5.7) that, as $j \to \infty$, $j/k \to 1$. Hence we may infer that, also as $j \to \infty$,

$$\frac{A_1(j) + O([\lambda_2/\lambda_1]^j)}{A_1(k) + O([\lambda_2/\lambda_1]^k)} \to 1.$$

Hence (5.7) can be rewritten:

$$x_m - y_u = (k-j)\gamma + o(1), \tag{5.8}$$

as $j \to \infty$ through the set $\mathfrak{J}(u)$.

From (5.8) we are forced to make conclusions: (i) that $x_m - y_u$ must be some fixed integral multiple of γ, say $x_m - y_u = r_u\gamma$ (for integer r_u); (ii) that for large j in $\mathfrak{J}(u)$ it must be true that $k-j = r_u$ exactly. Therefore, for all large j in $\mathfrak{J}(u)$ we may deduce from (5.7) the equation

$$A_1(j) + O([\lambda_2/\lambda_1]^j) = A_1(j+r_u) + O([\lambda_2/\lambda_1]^{j+r_u}). \tag{5.9}$$

Since $x_m > y_u$ it is necessary that $r_u > 0$. But (5.9) shows that $A_1(j) -$

$A_1(j+r_u)$ is a polynomial in j which tends to zero as $j \to \infty$ through values in $\mathfrak{J}(u)$. This can only be the case if $A_1(x) - A_1(x+r_u)$ is the zero polynomial, which requires that $A_1(x) = C_1$, say, a constant. Evidently we must have $C_1 > 0$ if $\sigma_m \geq 0$ for all m.

Thus we can rewrite (5.7) in more detail and deduce that for $j \varepsilon \mathfrak{J}(u)$,

$$\lambda_2^j A_2(j) + \lambda_3^j A(j) + \cdots$$
$$= \lambda_1^{-r_u}\{\lambda_2^{j+r_u} A_2(j+r_u) + \lambda_3^{j+r_u} A_3(j+r_u) + \cdots$$

This implies that, for all $j \varepsilon \mathfrak{J}(u)$,

$$\frac{A_2(j) + O([\lambda_3/\lambda_2]^j)}{A_2(j+r_u) + O([\lambda_3/\lambda_2]^{j+r_u})} = (\lambda_2/\lambda_1)^{r_u} . \tag{5.10}$$

By the nature of $A_2(j)$, there will be an integer $d_2 \geq 0$ such that

$$\frac{A_2(j)}{j^{d_2}} \sim \Phi_2(j), \text{ say,}$$

where $\Phi_2(j)$ is a bounded periodic or, possibly, almost-periodic, function. From (5.10) we can therfore claim that, as $j \to \infty$ through values in $\mathfrak{J}(u)$,

$$\left|\frac{\Phi_2(j) + o(1)}{\Phi_2(j+r_u) + o(1)}\right| \to (\lambda_2/\lambda_1)^{r_u}. \tag{5.11}$$

We introduced an infinite set $\mathfrak{J}$ earlier, of which $\mathfrak{J}(u)$ is an infinite subset, but left $\mathfrak{J}$ undefined. Let us therefore now define $\mathfrak{J}$ as consisting of any subsequence $\{j_\nu\}$ such that

$$|\Phi_2(j_\nu)| \to \limsup_{j\to\infty} |\Phi_2(j)| = \mu, \text{say,}$$

where the supremum refers to j increasing through integer values.

Suppose $\mu > 0$. Then, as $j \to \infty$ through the set $\mathfrak{J}(u)$

$$\liminf_{j \to \infty} |\Phi_2(j)| = \mu$$

and

$$\limsup_{j \to \infty} |\Phi_2(j + r_u)| \leq \mu .$$

Therefore

$$\liminf_{j \to \infty} \left| \frac{\Phi_2(j) + o(1)}{\Phi_2(j + r_u) + o(1)} \right| \geq 1. \tag{5.12}$$

However, $\lambda_2 < \lambda_1$ and $r_u > 0$, so $(\lambda_2/\lambda_1)^r < 1$. Thus (5.12) contradicts (5.11) and we are forced to conclude that $\mu = 0$. In other words, if

$$A_2(j) = j^{d_2}\Phi_2(j) + \text{terms involving lower powers of } j,$$

then $\Phi_2(j) \equiv 0$. This forces the conclusion that $A_2(j) \equiv 0$, and hence similar conclusions for $A_3(j)$, $A_4(j)$, and so on. Thus $\sigma_j = C_1\lambda_1^j$ for all j; and since we require $\sigma_0 = 1$, it follows that $C_1 = 1$ and that the $\{X_n\}$ are generated by the first-order Markov equation

$$X_n - \lambda_1 X_{n-1} = \epsilon_n .$$

Subject to Assumption A this is the only autoregressive model which allows both F and G to be in $\mathcal{N}_1$.

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