

SOME PROPERTIES OF BIVARIATE EMPIRICAL HAZARD
PROCESSES UNDER RANDOM CENSORING

by

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ABSTRACT

In Campbell (1982) and Campbell & Földes (1982) some asymptotic properties of bivariate empirical hazard processes under random censoring are given. Taking the representation of the empirical hazard process for bivariate randomly censored samples in Campbell (1982) as a starting point and restricting attention to strong properties, we obtain a speed of strong convergence for the weighted bivariate empirical hazard processes as well as a speed of strong uniform convergence for bivariate hazard rate estimators. Our approach is based on a local fluctuation inequality for the bivariate empirical hazard process and differs from the martingale methods quite often used in the univariate case.

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1. INTRODUCTION AND NOTATION

Martingale methods provide an elegant and powerful tool to derive strong and weak convergence properties of the empirical hazard process and related processes based on censored random samples; see e.g. Aalen (1978 a,b), Gill (1980,1983) and Shorack & Wellner (1986, Chapter 7). Questions concerning speed of almost sure convergence or extension to multivariate censored observations, however, don't seem to be yet considered along these lines. Since the empirical hazard process is composed of an ordinary and compound empirical process, a direct alternative approach might be patterned on well-known properties of these processes. Of particular importance are the local fluctuation inequalities available for both ordinary multivariate empirical processes (see e.g. Einmahl (1987)) and compound multivariate empirical processes that are allowed to have random jumps (see Einmahl & Ruymgaart (1986)). In Section 2 we exploit these inequalities to obtain a local fluctuation inequality for the multivariate empirical hazard process; for notational convenience we will restrict attention to the bivariate case.

Local fluctuation inequalities of this kind typically play a fundamental role in the study of global weak and strong asymptotic properties of the processes involved. In Section 3 we apply this inequality to obtain a speed of strong convergence for the *weighted* bivariate empirical hazard process; we briefly sketch, moreover, how a speed of strong uniform convergence of bivariate hazard *rate* estimators may be obtained. In the univariate case the speed of convergence of hazard rate and related estimators can be found in Schäfer (1986), where a related method is used. A classical approach to weak convergence of the empirical hazard process is given in Efron (1967) and Breslow & Crowley (1974). The initiating paper on this subject is Kaplan & Meier (1958); see also Meier (1975). In the bivariate case some asymptotic properties are obtained in Campbell (1981, 1982) and Campbell & Földes (1982). A practical motivation for the study of censoring in the multivariate case is contained in Campbell (1982).

Let $(X_{11}, X_{12}), \dots, (X_{n1}, X_{n2})$ and $(Y_{11}, Y_{12}), \dots, (Y_{n1}, Y_{n2})$ be two mutually independent sets of i.i.d. random vectors with values in $[0, \infty) \times [0, \infty)$, let F be the common d.f. of the (X_{i1}, X_{i2}) and G the common d.f. of the (Y_{i1}, Y_{i2}) . Under censoring the random vectors

$$(1.1) \quad (Z_{i1}, Z_{i2}) = (X_{i1} \wedge Y_{i1}, X_{i2} \wedge Y_{i2}), (\delta_{i1}, \delta_{i2}) = \left[1_{\{X_{i1} \leq Y_{i1}\}}, 1_{\{X_{i2} \leq Y_{i2}\}} \right],$$

are observed rather than the (X_{i1}, X_{i2}) . Note that the $(Z_{i1}, Z_{i2}, \delta_{i1}, \delta_{i2})$ are i.i.d.

It is convenient to write, for $(t_1, t_2) \in [0, \infty) \times [0, \infty)$,

$$(1.2) \quad F(\bar{t}_1, \bar{t}_2) = P(X_{i1} > t_1, X_{i2} > t_2),$$

$$(1.3) \quad H(t_1, t_2) = P(Z_{i1} \leq t_1, Z_{i2} \leq t_2), \quad H(\bar{t}_1, \bar{t}_2) = P(Z_{i1} > t_1, Z_{i2} > t_2),$$

$$(1.4) \quad H_{1,2}(t_1, t_2) = P(Z_{i1} \leq t_1) \cdot P(Z_{i2} \leq t_2),$$

$$(1.5) \quad H_j^*(t_1, t_2) = P(Z_{i1} \leq t_1, Z_{i2} \leq t_2, \delta_{ij} = 1), \quad j \in \{1, 2\},$$

$$(1.6) \quad H_1^*(t_1, \bar{t}_2) = P(Z_{i1} \leq t_1, Z_{i2} > t_2, \delta_{i1} = 1),$$

$$(1.7) \quad H_2^*(\bar{t}_1, t_2) = P(Z_{i1} > t_1, Z_{i2} \leq t_2, \delta_{i2} = 1).$$

The empirical analogues are

$$(1.8) \quad \hat{H}_n(t_1, t_2) = n^{-1} \sum_{i=1}^n 1_{[0, t_1] \times [0, t_2]}(Z_{i1}, Z_{i2}),$$

$$(1.9) \quad \hat{H}_{jn}^*(t_1, t_2) = n^{-1} \sum_{i=1}^n \delta_{ij} 1_{[0, t_1] \times [0, t_2]}(Z_{i1}, Z_{i2}), \quad j \in \{1, 2\},$$

with $\hat{H}_n(\bar{t}_1, \bar{t}_2)$, $\hat{H}_{1n}^*(t_1, \bar{t}_2)$ and $\hat{H}_{2n}^*(\bar{t}_1, t_2)$ defined in the obvious way. Along with the ordinary empirical process

$$(1.10) \quad U_n(t_1, t_2) = n^{\frac{1}{2}}(\hat{H}_n(t_1, t_2) - H(t_1, t_2)), \quad (t_1, t_2) \in [0, \infty) \times [0, \infty),$$

the pair of compound empirical processes ($j = 1, 2$)

$$(1.11) \quad U_{jn}^*(t_1, t_2) = n^{\frac{1}{2}}(\hat{H}_{jn}^*(t_1, t_2) - H_j^*(t_1, t_2)), \quad (t_1, t_2) \in [0, \infty) \times [0, \infty),$$

plays an important role. The notation $U_n(\bar{t}_1, \bar{t}_2)$, $U_{1n}^*(t_1, \bar{t}_2)$ and $U_{2n}^*(\bar{t}_1, t_2)$ is defined similarly. For any $a = (a_1, a_2)$, $b = (b_1, b_2) \in \mathbb{R}^2$ with $a_1 < b_1$, $a_2 < b_2$ we write $(a, b] = (a_1, b_1] \times (a_2, b_2]$, and similarly $[a, b)$, etc. The origin in \mathbb{R}^2 is written $O = (0, 0)$.

ASSUMPTION 1.1 The point $T = (T_1, T_2) \in (0, \infty) \times (0, \infty)$ is chosen such that $H(\bar{T}_1, \bar{T}_2) > 0$. All functions in (1.2)–(1.7) are supposed to be *continuous*. Because $H_j^* \leq H$ we have $H_j^* \ll H$. It will be assumed that $H \ll H_{1,2}$ and that

$$(1.12) \quad 0 < m_j^* = \text{ess inf}_{t \in [O, T]} \frac{dH_j^*}{dH}(t), \quad j \in \{1, 2\};$$

$$(1.13) \quad M = \text{ess sup}_{t \in [O, T]} \frac{dH}{dH_{1,2}}(t) < \infty.$$

(Note that (1.13) is trivially fulfilled in the univariate case.)

We will focus on estimation of the cumulative hazard function defined by

$$(1.14) \quad R(t_1, t_2) = -\log F(\bar{t}_1, \bar{t}_2), \quad (t_1, t_2) \in [0, \infty) \times [0, \infty).$$

Under the conditions mentioned above the hazard gradient approach of Marshall (1975) applies. Using the path-independence of line integrals (see e.g. Apostol (1957, Theorem 10–37)), Campbell (1982) arrives at the expression

$$(1.15) \quad R(t_1, t_2) = \int_{0 \leq u \leq t_1} \frac{1}{H(\bar{u}, \bar{0})} dH_1^*(u, \bar{0}) + \int_{0 \leq v \leq t_2} \frac{1}{H(\bar{t}_1, \bar{v})} dH_2^*(\bar{t}_1, v),$$

and its empirical analogue

$$(1.16) \quad \hat{R}_n(t_1, t_2) = \int_{0 \leq u \leq t_1} \frac{1}{\hat{H}_n(\bar{u}, \bar{0})} d\hat{H}_{1n}^*(u, \bar{0}) + \int_{0 \leq v \leq t_2} \frac{1}{\hat{H}_n(\bar{t}_1, \bar{v})} d\hat{H}_{2n}^*(\bar{t}_1, v).$$

Properties of the bivariate empirical hazard process

$$(1.17) \quad W_n(t) = W_n(t_1, t_2) = n^{\frac{1}{2}}(\hat{R}_n(t_1, t_2) - R(t_1, t_2)), (t_1, t_2) = t \in [0, T],$$

will be studied.

2. LOCAL FLUCTUATION INEQUALITY

Given any (random) function $L : \mathbb{R}^2 \rightarrow [-\infty, \infty]$ it is convenient to write

$$(2.1) \quad L\{(a, b)\} = L(b_1, b_2) - L(a_1, b_2) + L(a_1, a_2) - L(b_1, a_2).$$

The function

$$(2.2) \quad \psi(\lambda) = 2\lambda^{-2} \int_0^\lambda \log(1+x) dx, \quad \lambda > 0; \quad \psi(0) = 1,$$

will occur in the exponential probability bound. This function is continuous on $[0, \infty)$ and $\psi(\lambda) \downarrow 0$ as $\lambda \uparrow \infty$. We have, moreover, the useful property

$$(2.3) \quad \psi(\lambda) \geq \epsilon \psi(\epsilon\lambda), \quad 0 \leq \epsilon \leq 1, \quad \lambda \geq 0.$$

Throughout the remainder part of this paper the numbers $A, B, C \in (0, \infty)$ denote *generic constants* that are independent of all the relevant parameters, like in particular the sample size n . Furthermore let

$$(2.4) \quad \Psi_n(\lambda) = \min\{n^{-1}, \exp(-n\lambda)\}, \quad \lambda \geq 0.$$

Although the processes U_n and U_{jn}^* are not transformed to the unit interval it is clear (see e.g. Einmahl (1987, Section 6.3.c)) that the fluctuation inequalities in Einmahl (1987, Inequality 2.5) and Einmahl & Ruymgaart (1986, Theorem 1.1) remain true in the present situation, so that for arbitrary $(a,b) \subset [0,\infty) \times [0,\infty)$, $n \in \mathbb{N}$ and $\lambda > 0$ we have

$$(2.5) \quad \begin{aligned} & P(\sup_{(\alpha,\beta) \subset (a,b)} |U_n\{(\alpha,\beta)\}| \geq \lambda) \leq \\ & \leq C \exp \left[\frac{-A\lambda^2}{H\{(a,b)\}} \psi \left[\frac{B\lambda}{n^{\frac{1}{2}} H\{(a,b)\}} \right] \right], \end{aligned}$$

$$(2.6) \quad \begin{aligned} & P(\sup_{(\alpha,\beta) \subset (a,b)} |U_{jn}^*\{(\alpha,\beta)\}| \geq \lambda) \leq \\ & \leq C \exp \left[\frac{-A\lambda^2}{H_j^*\{(a,b)\}} \psi \left[\frac{B\lambda}{n^{\frac{1}{2}} H_j^*\{(a,b)\}} \right] \right], \quad j \in \{1,2\}. \end{aligned}$$

Let us note that (cf. Einmahl & Ruymgaart (1986, formula (1.8)))

$$(2.7) \quad \begin{aligned} & E \delta_{ij}^2 1_{[0,t_1] \times [0,t_2]}(Z_{i1}, Z_{i2}) = E \delta_{ij} 1_{[0,t_1] \times [0,t_2]}(Z_{i1}, Z_{i2}) = \\ & = H_j^*(t), \quad (t_1, t_2) = t \in [0,\infty) \times [0,\infty), \quad j \in \{1,2\}. \end{aligned}$$

The right-hand sides in (2.5) and (2.6) are to be understood as 0 in case $H\{(a,b)\} = 0$ respectively $H_j^*\{(a,b)\} = 0$.

THEOREM 2.1: local fluctuation inequality. *Let Assumption 1.1 be satisfied. For all $(a,b) \subset [0,T]$, $n \in \mathbb{N}$ and $\lambda \geq 0$ we have*

$$(2.8) \quad \begin{aligned} & P(\sup_{(\alpha,\beta) \subset (a,b)} |W_n\{(\alpha,\beta)\}| \geq \lambda) \leq \\ & \leq C \exp \left[\frac{-A\lambda^2}{H_{1,2}\{(a,b)\}} \psi \left[\frac{B\lambda}{n^{\frac{1}{2}} H_{1,2}\{(a,b)\}} \right] \right] + C \Psi_n(A H\{(a,b)\}). \end{aligned}$$

where the generic constants A, B, C depend only on M, m_1^*, m_2^* , and T.

PROOF. It is clear from (1.15), (1.16) and (2.1) that $W_n\{(\alpha, \beta)\} = \sum_{j=1}^6 W_{nj}(\alpha, \beta)$, where

$$(2.9) \quad W_{n1}(\alpha, \beta) = \int_{\alpha_2 < v \leq \beta_2} \frac{U_n(\bar{\alpha}_1, \bar{v}) - U_n(\bar{\beta}_1, \bar{v})}{H(\bar{\alpha}_1, \bar{v}) \hat{H}_n(\bar{\alpha}, \bar{v})} d \hat{H}_{2n}^*(\bar{\beta}_1, v),$$

$$(2.10) \quad W_{n2}(\alpha, \beta) = \int_{\alpha_2 < v \leq \beta_2} \frac{U_n(\bar{\beta}_1, \bar{v}) \{H(\bar{\beta}_1, \bar{v}) - H(\bar{\alpha}_1, \bar{v})\}}{H(\bar{\alpha}_1, \bar{v}) \hat{H}_n(\bar{\alpha}_1, \bar{v}) H(\bar{\beta}_1, \bar{v})} d \hat{H}_{2n}^*(\bar{\beta}_1, v),$$

$$(2.11) \quad W_{n3}(\alpha, \beta) = \int_{\alpha_2 < v \leq \beta_2} \frac{U_n(\bar{\beta}_1, \bar{v}) \{\hat{H}_n(\bar{\beta}_1, \bar{v}) - \hat{H}_n(\bar{\alpha}_1, \bar{v})\}}{\hat{H}_n(\bar{\alpha}_1, \bar{v}) H(\bar{\beta}_1, \bar{v}) \hat{H}_n(\bar{\beta}_1, \bar{v})} d \hat{H}_{2n}^*(\bar{\beta}_1, v),$$

$$(2.12) \quad W_{n4}(\alpha, \beta) = \int_{\alpha_2 < v \leq \beta_2} \frac{H(\bar{\alpha}_1, \bar{v}) - H(\bar{\beta}_1, \bar{v})}{H(\bar{\alpha}_1, \bar{v}) H(\bar{\beta}_1, \bar{v})} d U_{2n}^*(\bar{\beta}_1, v),$$

$$(2.13) \quad W_{n5}(\alpha, \beta) = \int_{\alpha_2 < v \leq \beta_2} \frac{1}{H(\bar{\alpha}_1, \bar{v})} d \{U_{2n}^*(\bar{\beta}_1, v) - U_{2n}^*(\bar{\alpha}_1, v)\},$$

$$(2.14) \quad W_{n6}(\alpha, \beta) = \int_{\alpha_2 < v \leq \beta_2} \frac{U_n(\bar{\alpha}_1, \bar{v})}{H(\bar{\alpha}_1, \bar{v}) \hat{H}_n(\bar{\alpha}_1, \bar{v})} d \{\hat{H}_{2n}^*(\bar{\alpha}_1, v) - \hat{H}_{2n}^*(\bar{\beta}_1, v)\}.$$

For $0 < \epsilon < 1$ let us single out the subsets

$$(2.15) \quad \Omega_{1n} = \{\hat{H}_n(\mathbb{T}_1, \mathbb{T}_2) \geq (1-\epsilon) H(\mathbb{T}_1, \mathbb{T}_2)\},$$

$$(2.16) \quad \Omega_{2n} = \{\hat{H}_n\{(a_1, b_1] \times [0, \infty)\} \leq (1+\epsilon) H\{(a_1, b_1] \times [0, \infty)\}\},$$

$$(2.17) \quad \Omega_{3n} = \{\hat{H}_{2n}^*\{(a_1, b_1] \times (a_2, b_2]\} \leq (1+\epsilon) H_2^*\{(a_1, b_1] \times (a_2, b_2]\}\},$$

$$(2.18) \quad \Omega_{4n} = \{\hat{H}_{2n}^*\{[0, \infty) \times (a_2, b_2]\} \leq (1+\epsilon) H_2^*\{[0, \infty) \times (a_2, b_2]\}\},$$

and let us introduce

$$(2.19) \quad \Omega_n = \bigcap_{k=1}^4 \Omega_{kn}.$$

In order to find upper bounds for the $P(\Omega_{kn}^c)$ it turns out that for intervals $(a,b]$ having not too small an H or H_j^* mass, (2.5) or (2.6) gives a suitable result. For small masses of order $1/n$, however, a better upper bound is obtained by simply applying Chebyshev's inequality.

Let us choose Ω_{3n}^c as an example and note that $P(\Omega_{3n}^c) \leq C \exp(-A n H_2^*\{(a,b)\})$ according to (2.6); by Chebyshev's inequality we have $P(\Omega_{3n}^c) \leq C H_2^*\{(a,b)\}$. Hence $P(\Omega_{3n}^c) \leq \min\{C H_2^*\{(a,b)\}, C \exp(-A n H_2^*\{(a,b)\})\}$ for a proper choice of the generic constants. It is easily seen that the upper bounds for $k \neq 3$ are essentially of smaller order so that, using (1.12), we arrive at

$$(2.20) \quad P(\Omega_n^c) \leq C \Psi_n(A H\{(a,b)\}).$$

Let us now consider the r.v.'s

$$\tilde{W}_{nj}(\alpha, \beta) = 1_{\Omega_n} W_{nj}(\alpha, \beta).$$

It is obvious that (2.8) is proved when it can be shown that each of the probabilities $P(\sup_{(\alpha, \beta] \subset (a, b]} |\tilde{W}_{nj}(\alpha, \beta)| \geq \lambda)$ is bounded above by the first term on the right in (2.8) for proper choices of the generic constants; note that the second term on the right in (2.8) is accounted for by (2.20). Since these r.v.'s can be dealt with in more or less the same way let us just select two typical examples.

Let us first take $j=3$ and note that application of (1.12) yields

$$\begin{aligned}
 (2.21) \quad \sup_{(\alpha, \beta] \subset (a, b]} |\tilde{W}_{n3}(\alpha, \beta)| &\leq (1-\epsilon)^{-2} (H(\mathbb{T}_1, \mathbb{T}_2))^{-3} (1+\epsilon) H\{(a_1, b_1] \times [0, \infty)\} \\
 &\times (1+\epsilon) H_2^*\{[0, \infty) \times (a_2, b_2]\} \sup_{(\alpha, \beta] \subset [0, \infty) \times [0, \infty)} |U_n\{(\alpha, \beta)\}| \leq \\
 &\leq C H_{1,2}\{(a, b)\} \sup_{(\alpha, \beta] \subset [0, \infty) \times [0, \infty)} |U_n\{(\alpha, \beta)\}|.
 \end{aligned}$$

Application of (2.5) with $(a, b]$ replaced by $[0, \infty) \times [0, \infty)$ yields

$$\begin{aligned}
 (2.22) \quad P(\sup_{(\alpha, \beta] \subset (a, b]} |\tilde{W}_{nj}(\alpha, \beta)| \geq \lambda) &\leq \\
 &\leq C \exp\left[\frac{-A\lambda^2}{H_{1,2}\{(a, b)\}^2} \psi\left[\frac{B\lambda}{n^{\frac{1}{2}} H_{1,2}\{(a, b)\}}\right]\right] \leq \\
 &\leq C \exp\left[\frac{-A\lambda^2}{H_{1,2}\{(a, b)\}} \psi\left[\frac{B\lambda}{n^{\frac{1}{2}} H_{1,2}\{(a, b)\}}\right]\right].
 \end{aligned}$$

Hence the right-hand side of (2.22) is of the required order of magnitude.

As a second example let us take $j=5$; in this case we don't need to restrict the r.v. to Ω_n so that we will consider $W_{n5}(\alpha, \beta)$. Partial integration yields that $W_{n5}(\alpha, \beta) =$

$\sum_{j=1}^3 W_{n5}^{(j)}(\alpha, \beta)$, where

$$(2.23) \quad W_{n5}^{(1)}(\alpha, \beta) = \frac{U_{2n}^*(\beta_1, \beta_2) - U_{2n}^*(\bar{\alpha}_1, \beta_2) - U_{2n}^*(\beta_1, \alpha_2) + U_{2n}^*(\bar{\alpha}_1, \alpha_2)}{H(\bar{\alpha}_1, \beta_2)},$$

$$(2.24) \quad W_{n5}^{(2)}(\alpha, \beta) = \{U_{2n}^*(\beta_1, \alpha_2) - U_{2n}^*(\bar{\alpha}_1, \alpha_2)\} \frac{H(\bar{\alpha}_1, \bar{\alpha}_2) - H(\bar{\alpha}_1, \beta_2)}{H(\bar{\alpha}_1, \bar{\alpha}_2)H(\bar{\alpha}_1, \beta_2)},$$

$$(2.25) \quad W_{n5}^{(3)}(\alpha, \beta) = \int_{\alpha_2 < v \leq \beta_2} \{U_{2n}^*(\beta_1, v) - U_{2n}^*(\bar{\alpha}_1, v)\} d\left[\frac{1}{H(\bar{\alpha}_1, v)}\right],$$

We can simply handle $W_{n5}^{(3)}$ by noting that $\frac{1}{H(\bar{\alpha}_1, v)} \uparrow$ as $v \uparrow$.

Restricting ourselves to $W_{n5}^{(1)}(\alpha, \beta)$ we observe that

$$(2.26) \quad \sup_{(\alpha, \beta] \subset (a, b]} |W_{n5}^{(1)}(\alpha, \beta)| \leq (H(\mathbb{T}_1, \mathbb{T}_2))^{-1} \sup_{(\alpha, \beta] \subset (a, b]} |U_{2n}^*\{(\alpha, \beta)\}| .$$

Applying (2.6), the fact that $H_2^*\{(a, b)\} \leq H\{(a, b)\}$ along with (1.13), and (2.3) with $\epsilon = H_2^*\{(a, b)\}/(MH_{1,2}\{(a, b)\})$ we find

$$(2.27) \quad \begin{aligned} P(\sup_{(\alpha, \beta] \subset (a, b]} |W_{n5}^{(1)}| \geq \lambda) &\leq \\ &\leq C \exp \left[\frac{-A\lambda^2}{H_2^*\{(a, b)\}} \psi \left[\frac{B\lambda}{n^{\frac{1}{2}} H_2^*\{(a, b)\}} \right] \right] \leq \\ &\leq C \exp \left[\frac{-A\lambda^2}{H_{1,2}\{(a, b)\}} \psi \left[\frac{B\lambda}{n^{\frac{1}{2}} H_{1,2}\{(a, b)\}} \right] \right], \end{aligned}$$

which is of the desired order. Q.E.D.

The theorem will now be applied to obtain inequalities for the local behavior of weighted bivariate empirical hazard processes. Under the present conditions it is natural to weight the hazard process by an appropriate function of $H_{1,2}$, the product of the marginals of H ; we will consider

$$(2.28) \quad W_n(t)/H_{1,2}(t)^{\frac{1}{2}-\delta}, \quad t \in [0, T], \quad 0 \leq \delta \leq \frac{1}{2} .$$

In most applications the intervals $(a, b]$ over which the local properties are considered, will arise as elements of a partition. In such cases the points a and b will be close together.

A reasonable condition turns out to be

$$(2.29) \quad 0 < \frac{1}{4} H_{1,2}(b) \leq H_{1,2}(a) < H_{1,2}(b) .$$

THEOREM 2.2: local behavior weighted processes. *Let Assumption 1.1 be fulfilled. For any $n \in \mathbb{N}$, $\lambda \geq 0$, $0 \leq \delta \leq \frac{1}{2}$ and $(a, b] \subset [0, T]$ with a, b satisfying (2.29) we have*

$$(2.30) \quad \mathbb{P} \left[\sup_{t \in (a,b]} \frac{|W_n(t)|}{H_{1,2}(t)^{\frac{1}{2}-\delta}} \geq \lambda \right] \leq \\ \leq C \exp \left[\frac{-A\lambda^2}{H_{1,2}(b)^{2\delta}} \psi \left[\frac{B\lambda}{n^{\frac{1}{2}} H_{1,2}(b)^{\frac{1}{2}+\delta}} \right] \right] + C \Psi_n(AH(b)).$$

PROOF. The proof may be patterned on that of Ruymgaart & Wellner (1984, Corollary 2.1) or Einmahl & Ruymgaart (1986, Corollary 2.1). Q.E.D.

3. APPLICATIONS

In this section some global results will be briefly sketched by applying the local inequalities of the preceding section. As an application of Theorem 2.2 we have the following result on strong convergence of weighted processes.

Let Assumption 1.1 be fulfilled. Then there exists $K \in (1, \infty)$ such that

$$(3.1) \quad \limsup_{n \rightarrow \infty} \sup_{t \in [0, T]: H(t) \geq \frac{K \log n}{n}} \frac{|W_n(t)|}{(H_{1,2}(t) \log n)^{\frac{1}{2}}} \stackrel{\text{a.s.}}{\leq} K.$$

The proof is very similar to that of Ruymgaart & Wellner (1984, Theorem 3.1) or Einmahl & Ruymgaart (1986, Theorem 2.1). Let us for $N \in \mathbb{N}$ introduce the partition

$$(3.2) \quad \mathcal{P} = \left\{ \left[H_1^{-1} \left[\frac{(j-1)T_1}{N} \right], H_1^{-1} \left[\frac{j T_1}{N} \right] \right] \times \left[H_2^{-1} \left[\frac{(k-1)T_2}{N} \right], H_2^{-1} \left[\frac{k T_2}{N} \right] \right] : \right. \\ \left. j, k \in \{1, \dots, N\} \right\},$$

and, for $N = [4n/K] + 1$, consider the subfamily

$$(3.3) \quad \mathcal{P}_n = \{(a, b) \in \mathcal{P} : H(b) \geq (K \log n)/n\}.$$

It is easy to see that

$$(3.4) \quad \{t \in [0, T] : H(t) \geq (K \log n)/n\} \subset \cup_{(a,b) \in \mathcal{P}_n} (a,b] \subset \\ \subset \{t \in [0, T] : H(t) \geq (K \log n)/(2n)\} .$$

Because each $(a,b) \in \mathcal{P}$ satisfies (2.29), application of Theorem 2.2 to any $(a,b) \in \mathcal{P}_n$ leads to an upper bound that can be made suitably small by choosing K sufficiently large. Due to (1.13) we have, moreover, that $H_{1,2}(b) \geq H(b)/M \geq (K \log n)/(2n M)$. This implies that

$$(3.5) \quad \Psi_n(H(b)) \leq C \exp(-A K \log n) ,$$

which is sufficiently small for K sufficiently large.

Along similar lines weak convergence of the weighted bivariate hazard process might be considered. We will, however, rather consider an application of Theorem 2.1 to the estimation of the hazard rate, under suitable smoothness conditions defined by

$$(3.6) \quad \rho(t) = \partial^2 R(t_1, t_2) / \partial t_1 \partial t_2 .$$

We restrict this discussion to naïve estimators of the form

$$(3.7) \quad \hat{\rho}_n(t) = (\gamma_n)^{-1} \hat{R}_n\{I_{t,n}\} ,$$

where for suitable $\gamma(n) \in (0, \infty)$

$$(3.8) \quad I_{t,n} = (t_1 - \frac{1}{2} \gamma_n^{\frac{1}{2}}, t_1 + \frac{1}{2} \gamma_n^{\frac{1}{2}}] \times (t_2 - \frac{1}{2} \gamma_n^{\frac{1}{2}}, t_2 + \frac{1}{2} \gamma_n^{\frac{1}{2}}] .$$

Next let us introduce

$$(3.9) \quad \rho_n(t) = (\gamma_n)^{-1} R\{I_{t,n}\} .$$

It is tacitly understood that in the above expressions we take $R(t) = \hat{R}_n(t) = 0$ for

$t \notin [0, \infty) \times [0, \infty)$. Since it is well-known that the speed of convergence to 0 of the non-random part $\rho_n - \rho$ depends on the smoothness of ρ , we will only consider the random part $\hat{\rho}_n - \rho_n$. Let Λ denote Lebesgue measure in \mathbb{R}^2 . Let us assume that $H_{1,2} \ll \Lambda$ so that $H \ll \Lambda$, in view of Assumption 1.1 and that, for some $S \in [0, T)$,

$$(3.10) \quad \text{ess sup}_{t \in [S, T]} \frac{dH_{1,2}}{d\Lambda}(t) < \infty;$$

$$(3.11) \quad 0 < \text{ess inf}_{t \in [S, T]} \frac{dH}{d\Lambda}(t).$$

We will, moreover, need that

$$(3.12) \quad \gamma_n \rightarrow 0, \quad n \gamma_n / \log n \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$

If in addition to Assumption 1.1 conditions (3.10)–(3.12) are fulfilled there exists $K \in (0, \infty)$ such that

$$(3.13) \quad \limsup_{n \rightarrow \infty} \left[\frac{n \gamma_n}{\log n} \right]^{\frac{1}{2}} \sup_{t \in (S, T]} |\hat{\rho}_n(t) - \rho_n(t)| \stackrel{\text{a.s.}}{\leq} K.$$

The proof follows as usual by partitioning $(S, T]$ into rectangles with sides of order $\gamma_n^{-\frac{1}{2}}$ in such a way that any square in (3.8) intersects at most 4 adjacent rectangles in the partition. Denoting this partition by \mathcal{P}_n we have

$$(3.14) \quad \left[\frac{n \gamma_n}{\log n} \right]^{\frac{1}{2}} \sup_{t \in (S, T]} |\hat{\rho}_n(t) - \rho_n(t)| \leq \\ \leq 4(\gamma_n \log n)^{-\frac{1}{2}} \max_{(a, b] \in \mathcal{P}_n} \sup_{(\alpha, \beta] \subset (a, b]} |W_n\{(\alpha, \beta]\}|.$$

Due to (3.10) the $H_{1,2}$ mass of each $(a, b] \in \mathcal{P}_n$ is uniformly of order γ_n . By condition (3.11) the numbers $\Psi_n(A \cap H\{(a, b]\})$ are uniformly of the right order as well.

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REFERENCES

- [1] AALEN, O. (1978a). Nonparametric inference for a family of counting processes. *Ann. Statist.* 6, 701–726.
- [2] AALEN, O. (1978b). Weak convergence of stochastic integrals related to counting processes. *Z. Wahrsch. Verw. Gebiete* 38, 261–277.
- [3] APOSTOL, T.M. (1957). *Mathematical Analysis*. Addison–Wesley, Reading, Mass.
- [4] BRESLOW, N. & CROWLEY, J. (1974). A large sample study of the life table and product limit estimates under random censorship. *Ann. Statist.* 2, 437–453.
- [5] CAMPBELL, G. (1981). Nonparametric bivariate estimation with randomly censored data. *Biometrika* 68, 417–422.
- [6] CAMPBELL, G. (1982). Asymptotic properties of several nonparametric multivariate distribution function estimators under random censoring. *IMS Lecture Notes – Mon. Ser.* 2, 243–256. IMS, Hayward.
- [7] CAMPBELL, G. & FOLDES, A. (1982). Large–sample properties of nonparametric bivariate estimators with censored data. In: *Proc. Int. Coll. Nonparametric Statist. Inf., 1980*, North–Holland, New York.
- [8] EFRON, B. (1967). The two–sample problem with censored data. *Proc. Fifth Berkeley Symp. Math. Statist. Probability* 4, 831–853. Un. California Press, Berkeley.
- [9] EINMAHL, J.H.J. (1987). *Multivariate Empirical Processes*. CWI Tract 32, Amsterdam.

- [10] EINMAHL, J.H.J. & RUYMGAART, F.H. (1986). Some properties of weighted compound multivariate empirical processes. *Sankhyā A* 48, 393-403.
- [11] GILL, R.D. (1980). *Censoring and Stochastic Integrals*. Math. Centre Tract 124, Amsterdam.
- [12] GILL, R.D. (1983). Large sample behavior of the product-limit estimator on the whole line. *Ann. Statist.* 11, 49-58.
- [13] KAPLAN, E.L. & MEIER, P. (1958). Nonparametric estimation from incomplete observations. *J. Am. Statist. Assoc.* 53, 457-481.
- [14] MARSHALL, A.W. (1975). Some comments on the hazard gradient. *Stoch. Proc. Appl.* 3, 293-300.
- [15] MEIER, P. (1975). Estimation of a distribution function from incomplete observations. In: *Perspectives in Statistics*, J. Gani, ed., 67-87. Ac. Press, London.
- [16] RUYMGAART, F.H. & WELLNER, J.A. (1984). Some properties of weighted multivariate empirical processes. *Statist. & Dec.* 2, 199-223.
- [17] SCHAFER, H. (1986). Local convergence of empirical measures in the random censorship situation with application to density and rate estimators. *Ann. Statist.* 14, 1240-1245.
- [18] SHORACK, G.R. & WELLNER, J.A. (1986). *Empirical Processes with Applications to Statistics*. Wiley, New York.

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