A Class of Multivariate Distributions Including the Multivariate Logistic*

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ABSTRACT

In this paper, a class of multivariate distributions including the multivariate logistic is defined and studied. Many properties of this class are obtained. Some sufficient conditions of this class are given.

Key words: Dirichlet distribution, multivariate logistic distribution, neutrality, spherical distribution.

*This work was supported by the Science Foundation of Academia Sinica.

1. Introduction.

The logistic distribution with density

$$(1.1)\qquad f(x) = \frac{e^{-x}}{(1+e^{-x})^2} \quad \text{for } -\infty < x < \infty$$

has been widely used by Berkson (1944), Berkson and Hodges (1960) as a model for analyzing bioassay and other experiments involving quantal response, by Pearl and Read (1920) in studies pertaining to population growth, and by Plackett (1959) in connection with problems involving censored data.

Gumbel (1961) proposed two bivariate logistic distributions with logistic marginals – type I being

$$(1.2)\qquad f(x,y) = \frac{2e^{-(x+y)}}{(1+e^{-x}+e^{-y})^2} \quad \text{for } -\infty < x, \quad y < \infty$$

and Type II belonging to the Morgenstern type. Gumbel Type I distribution appears to be a natural generalization of the univariate logistic distribution and it (the multivariate form) has been studied by Malik and Abraham (1973).

In this paper, a class of multivariate distributions including the Gumbel Type I distribution is considered, with the property that marginal distributions are of univariate form, and discussed on some distributional properties. In particular, sufficient conditions of this class are given.

2. Notation and Preliminary Results.

In this paper, we use the notation $X \sim N(\mu,\sigma^2)$ to indicate that X is a random variable distributed as $N(\mu,\sigma^2)$. If X and Y have the same distribution, we write $X \overset{d}{=} Y$. We call $\underset{\sim}{x}(n\times 1)$ having spherical distribution if $\underset{\sim}{x} \overset{d}{=} \underset{\sim}{\Gamma}\underset{\sim}{x}$ for each $\underset{\sim}{\Gamma} \in 0(n)$ (0(n) denotes the set of n×n orthogonal matrices) denoted by $\underset{\sim}{x} \sim S_n(\varphi)$, where $\varphi(t't)$ is $\underset{\sim}{x}$'s c.f. (characteristic function).

The following are noteworthy. We set these results to be used in the sequel.

2.1. Suppose $Z_1,...,Z_{m+1}$ are independent random variables having Gamma distributions $G(\alpha_1),...,G(\alpha_{m+1})$, where the density of G(u) is $f(x) = x^{u-1}e^{-x}/\Gamma(u)$ for $x > 0$ and $f(x) = 0$ otherwise. Let $Y_i = Z_i \,/\, (\sum_1^{m+1} Z_j)$ i=1,...,m. Then $(Y_1,...,Y_m)$ is called having m–variate Dirichlet distribution and denote by symbol $D(\alpha_1,...,\alpha_m;\, \alpha_{m+1})$ or $(Y_1,...,Y_m,\, Y_{m+1}) \sim D(\alpha_1,...,\alpha_m,\, \alpha_{m+1})$, where $Y_{m+1} = 1 - \sum_{i=1}^{m} Y_i$.

2.2. If $(Y_1,...,Y_m,\, Y_{m+1}) \sim D(\alpha_1,...,\alpha_{m+1})$, then the joint density of $(Y_1,...,Y_m)$ is given by

$$\frac{\Gamma\left(\sum_1^{m+1} \alpha_i\right)}{\prod_1^{m+1} \Gamma(\alpha_i)} \prod_{i=1}^{m} y_i^{\alpha_i - 1} \left(1 - \sum_{i=1}^{m} y_i\right)^{\alpha_{m+1}-1}, \quad y_i > 0,\ i=1,...,m,\ \sum_{i=1}^{m} y_i < 1, \tag{2.1}$$

where $\alpha_i > 0$, i=1,...,m+1.

2.3. If $(Y_1,...,Y_m) \sim D(\alpha_1,...,\alpha_m;\, \alpha_{m+1})$ and $r_1,...,r_t$ are integers such that $0 < r_1 < ... < r_t = m+1$, then

$$\left[\sum_{i=1}^{r_i} Y_i,\ \sum_{r_1+1}^{r_2} Y_i, ...,\ \sum_{r_{t-1}+1}^{r_t} Y_i\right] \sim D\left[\sum_1^{r_1} \alpha_i,\ \sum_{r_1+1}^{r_2} \alpha_i, ...,\ \sum_{r_{t-1}+1}^{r_t} \alpha_i\right].$$

3. Definition.

If $(Y_1,\dots,Y_m) \sim D(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$. Let

$$X_i = \log \frac{Y_{m+1}}{Y_i}, \quad i=1,\dots,m . \tag{3.1}$$

Then distribution of random vector $(X_1,\dots,X_m)$ is called DL–distribution and denoted $DL(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$.

According to definition, we may compute density function of random vector $(X_1,\dots,X_m)$. Since $(Y_1,\dots,Y_m) \sim D(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$ then the density of $(Y_1,\dots,Y_m)$ is (2.1). Let transformation (3.1), therefore

$$Y_i = \frac{e^{-x_i}}{1 + \sum_1^m e^{-x_j}}, \quad i=1,\dots,m$$

and the Jacobian of transformation from $(Y_1,\dots,Y_m)$ to $(X_1,\dots,X_m)$ is $e^{-\sum_1^m x_i} / (1 + \sum_1^m e^{-x_i})^{m+1}$. We obtain the joint density function of $(X_1,\dots,X_m)$ as follows:

$$\frac{\Gamma(\sum_1^{m+1} \alpha_i)}{\prod_1^{m+1} \Gamma(\alpha_i)} \frac{\exp(-\sum_{j=1}^m \alpha_j x_j)}{(1 + \sum_{j=1}^m e^{-x_j})^{\sum_1^{m+1} \alpha_i}} \tag{3.2}$$

$$\frac{\Gamma(\alpha)}{\prod_1^{m+1} \Gamma(\alpha_i)} \frac{\exp(-\sum_{j=1}^m \alpha_j x_j)}{(1 + \sum_{j=1}^m e^{-x_j})} \quad \text{for } -\infty < x_j < \infty, \quad j=1,\dots,m$$

where $\alpha = \alpha_1 + \dots + \alpha_{m+1}$, $\alpha_j > 0$, $j=1,\dots,m+1$.

When $\alpha_1 = \alpha_2 = \dots = \alpha_m = \alpha_{m+1} = 1$, (3.2) becomes the result of Malik and Abraham (1973) — multivariate logistic type I distribution.

4. Some Properties of $DL(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$.

Using definition and preliminary results in section 2, we get the following properties of $DL(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$.

Theorem 4.1. Suppose $Y_1,\dots,Y_{m+1}$ are independent random variables having Gamma distribution $G(\alpha_1),\dots,G(\alpha_{m+1})$. Let

$$X_i = \log(Y_{m+1}/Y_i), \quad i=1,\dots,m.$$

Then $(Y_1,\dots,Y_m) \sim D(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$.

Proof. Let

$$Z_i = Y_i \,/\, (Y_1 + \dots + Y_{m+1}), \quad i=1,\dots,m.$$

Then $(Z_1,\dots,Z_m)$ has the m–variate Dirichlet distribution $D(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$. According to definition of DL–distribution, we obtain that

$$(\log((1 - \sum_1^m Z_i)/Z_1),\dots, \log((1 - \sum_1^m Z_i)/Z_m) \sim DL(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1}).$$

However

$$\log((1 - \sum_1^m Z_i)/Z_i) = \log(Y_{m+1}/Y_i)$$

for i=1,...,m, therefore, $(X_1,\dots,X_m) \sim DL(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$.

Theorem 4.2. If m–variate random variable $(X_1,\dots,X_m)$ has DL–distribution $DL(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$, then the marginal distribution function of $(X_1,\dots,X_n)$ $(n < m)$ is that of a n–variate DL–distribution $DL(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$.

Corollary 1. If $(X_1,\dots,X_m) \sim D(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$, then $X_i \sim DL(\alpha_i;\ \alpha_{m+1})$ for i=1,...,m.

Corollary 2. If $(X_1,\dots,X_m) \sim D(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$, then $(X_1,\dots,X_{j-1},\ X_{j+1},\dots,X_m) \sim DL(\alpha_1,\dots,\alpha_{j-1},\ \alpha_{j+1},\dots,\alpha_m;\ \alpha_{m+1})$ for $j=1,\dots,m$.

The proofs of Theorem 4.2 and Corollaries are very easy, hence they are omitted.

Theorem 4.3. If $(X_1,\dots,X_m) \sim D(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$, then $(X_1-X_j, X_2-X_j,\dots,X_{j-1}-X_j, -X_j, X_{j+1}-X_j,\dots,X_m-X_j) \sim DL(\alpha_1,\dots,\alpha_{j-1},\alpha_{m+1},\alpha_{j+1},\dots,\alpha_m;\ \alpha_j)$.

Proof. Since $(X_1,\dots,X_m) \sim D(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$, then

$$(X_1,\dots,X_m) \overset{d}{=} (\log(Y_{m+1}/Y_1),\dots, \log(Y_{m+1}/Y_m)),$$

where Y_i has Gamma distribution $G(\alpha_i)$, $i=1,\dots,m+1$. Therefore,

$$(X_1-X_j,\dots,X_{j-1}-X_j,-X_j,X_{j+1}-X_j,\dots,X_m-X_j) \overset{d}{=} (\log(Y_j/Y_1),\dots,\log(Y_j/Y_{j-1}),$$

$$\log(Y_j/Y_{m+1}), \log(Y_j/Y_{j+1}),\dots,\log(Y_j/Y_m)) \sim DL(\alpha_1,\dots,\alpha_{j-1},\alpha_{m+1},$$

$$\alpha_{j+1},\dots,\alpha_m;\ \alpha_j)\ .$$

Theorem 4.4. If $(X_1,\dots,X_m) \sim D(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$. Let

$$Z_j = X_j + \log(1 + \sum_1^n e^{-x_i}),\quad j = n+1,\dots,m,$$

then $(Z_{n+1},\dots,Z_m) \sim DL(\alpha_{n+1},\dots,\alpha_m;\ \alpha_{m+1} + \sum_1^n \alpha_i)$.

Proof. Since $(X_1,\dots,X_m) \sim DL(\alpha_1,\dots,\alpha_m;\ \alpha_{m+1})$, then

$$(X_1,\dots,X_m) \overset{d}{=} (\log(Y_{m+1}/Y_1,\dots\log(Y_{m+1}/Y_m))$$

where $(Y_1,\dots,Y_m,\ Y_{m+1}) \sim D(\alpha_1,\dots,\alpha_m,\alpha_{m+1})$, $Y_{m+1} = 1 - \sum_1^m Y_i$. Therefore,

$$(Z_{n+1},\dots,Z_m) \overset{d}{=} (\log((Y_{m+1} + \sum_1^n Y_i)/Y_{n+1}),\dots,\log((Y_{m+1} + \sum_1^n Y_i)/Y_m))$$

$$= (\log((1-\sum_{n+1}^{m} Y_i)/Y_{n+1}),\ldots,\log((1-\sum_{n+1}^{m} Y_i)/Y_m))$$

$$\sim DL(\alpha_{n+1},\ldots,\alpha_m;\ \alpha_{m+1}+\sum_1^n \alpha_i)\ .$$

Corollary 1. If $(X_1,\ldots,X_m) \sim D(\alpha_1,\ldots,\alpha_m;\ \alpha_{m+1})$, then

$$X_m + \log(1+\sum_1^{m-1} e^{-x_i}) \sim DL(\alpha_m,\ \alpha-\alpha_m)$$

and is independent of $(X_1,\ldots,X_{m-1})$, where $\alpha = \sum_1^{m+1} \alpha_i$.

Theorem 4.5. If $(X_1,\ldots,X_m) \sim D(\alpha_1,\ldots,\alpha_m;\ \alpha_{m+1})$ and is distributed independently of $Y \sim DL(\alpha;\beta)$ and if $\alpha = \alpha_1 + \ldots + \alpha_{m+1}$, then

$$(X_1 + \log(1+e^Y(1+\sum_1^m e^{-x_i})),\ldots,X_m+\log(1+e^Y(1+\sum_1^m e^{-x_i})))$$

is distributed according to $DL(\alpha_1,\ldots,\alpha_m;\ \alpha_{m+1}+\beta)$.

The proof can be easily obtained by the method of computing density. Hence it is omitted.

Theorem 4.6. If $\underset{\sim}{x} \sim S_n(\varphi)$ (spherical distributions with characteristic function φ), partition $\underset{\sim}{x}$ as follows:

$$\underset{\sim}{x} = \begin{bmatrix} \underset{\sim}{x}^{(1)} \\ \vdots \\ \underset{\sim}{x}^{(m+1)} \end{bmatrix} \begin{matrix} n_1 \\ \vdots \\ n_{m+1} \end{matrix}$$

and $P(\underset{\sim}{x} = \underset{\sim}{0}) = 0$. Let

$$Y_i = \log(\underset{\sim}{x}^{(m+1)'}\underset{\sim}{x}^{(m+1)}/\underset{\sim}{x}^{(i)'}\underset{\sim}{x}^{(i)}),\quad i=1,\ldots,m.$$

Then $(Y_1,\ldots,Y_m) \sim DL(n_1/2,\ldots,n_m/2;\ n_{m+1}/2)$.

Proof. Since $\underset{\sim}{x} \sim S_n(\varphi)$ with $P(\underset{\sim}{x} = \underset{\sim}{0}) = 0$, therefore we have

$$(\underset{\sim}{x}^{(1)'}\underset{\sim}{x}^{(1)}/\underset{\sim}{x}'\underset{\sim}{x},\ldots,\underset{\sim}{x}^{(m)'}\underset{\sim}{x}^{(m)}/\underset{\sim}{x}'\underset{\sim}{x}) \sim D(n_1/2,\ldots,n_m/2;\, n_{m+1}/2).$$

Put

$$Y_i = \log(\underset{\sim}{x}^{(m+1)'}\underset{\sim}{x}^{(m+1)} \,/\, \underset{\sim}{x}^{(i)'}\underset{\sim}{x}^{(i)}), \quad i=1,\ldots,m.$$

According to definition of DL–distribution, we obtain the result, i.e., $(Y_1,\ldots,Y_m) \sim DL(n_1/2,\ldots,n_m/2;\, n_{m+1}/2)$.

Theorem 4.7. If $\underset{\sim}{x} \sim S_n(\varphi)$ with $P(\underset{\sim}{x} = \underset{\sim}{0}) = 0$, and $\underset{\sim}{A}_i$ is projection matrix with $\underset{\sim}{A}_i\underset{\sim}{A}_j = 0$ for any $j \neq i$, $i = 1,\ldots,m$. Let

$$Y_j = \log(\underset{\sim}{x}'\underset{\sim}{A}_{m+1}\underset{\sim}{x}/\underset{\sim}{x}'\underset{\sim}{A}_j\underset{\sim}{x}), \quad j=1,\ldots,m$$

where $\underset{\sim}{x}'\underset{\sim}{A}_{m+1}\underset{\sim}{x} = 1 - \sum_1^m \underset{\sim}{x}'\underset{\sim}{A}_j\underset{\sim}{x}$, then $(Y_1,\ldots,Y_m) \sim D(n_1/2,\ldots,n_m/2;\, n_{m+1}/2)$.

Proof. Since $\underset{\sim}{x} \sim S_n(\varphi)$ with $P(\underset{\sim}{x} = \underset{\sim}{0}) = 0$, then $(\underset{\sim}{x}'\underset{\sim}{A}_1\underset{\sim}{x},\ldots,\underset{\sim}{x}'\underset{\sim}{A}_m\underset{\sim}{x}) \overset{d}{=} R^2(d_1^2,\ldots,d_m^2)$, where $(d_1^2,\ldots,d_m^2) \sim D(n_1/2,\ldots,n_m/2;\, n_{m+1}/2)$. Therefore

$$\begin{aligned}(Y_1,\ldots,Y_m) &= (\log(\underset{\sim}{x}'\underset{\sim}{A}_{m+1}\underset{\sim}{x} \,/\, \underset{\sim}{x}'\underset{\sim}{A}_1\underset{\sim}{x}),\ldots,\log(\underset{\sim}{x}'\underset{\sim}{A}_{m+1}\underset{\sim}{x} \,/\, \underset{\sim}{x}'\underset{\sim}{A}_m\underset{\sim}{x})) \\ &\overset{d}{=} (\log((1 - \sum_1^m d_i^2)/d_1^2),\ldots,\log((1 - \sum_1^m d_i^2)/d_m^2)) \\ &\sim DL(n_1/2,\ldots,n_m/2;\, n_{m+1}/2).\end{aligned}$$

Theorem 4.8. If $(X_1,\ldots,X_m) \sim D(\alpha_1,\ldots,\alpha_m;\, \alpha_{m+1})$. Let

$$Z_j = -\log \sum_{r_{j-1}+1}^{r_j} e^{-x_i}, \quad j=1,\ldots,s$$

where $0 = r_0 < r_1 < \ldots < r_s = m$, then $(Z_1,\ldots,Z_s)$ has the s–variate DL–distribution

$$DL\left[\sum_{1}^{r_1}\alpha_j,\ \sum_{r_1+1}^{r_2}\alpha_j,\ \ldots,\ \sum_{r_{2-1}+1}^{m}\alpha_j;\ \alpha_{m+1}\right].$$

Proof. Since $(X_1,\ldots,X_m) \sim DL(\alpha_1,\ldots,\alpha_m;\ \alpha_{m+1})$, therefore

$$(X_1,\ldots,X_m) \stackrel{d}{=} (\log(Y_{m+1}/Y_1),\ \ldots,\ \log(Y_{m+1}/Y_m))$$

where $(Y_1,\ldots,Y_m)$ has the m–variate Dirichlet distribution $D(\alpha_1,\ldots,\alpha_m;\ \alpha_{m+1})$, $Y_{m+1} = 1-(Y_1+\ldots+Y_m)$. Then

$$(e^{-x_1},\ldots,e^{-x_m}) \stackrel{d}{=} (Y_1/Y_{m+1},\ldots,Y_m/Y_{m+1})$$

$$\left[\sum_{1}^{r_1} e^{-x_i},\ \sum_{r_1+1}^{r_2} e^{-x_i},\ \ldots,\ \sum_{r_{s-1}+1}^{m} e^{-x_i}\right] \stackrel{d}{=} \left[\sum_{1}^{r_1} Y_i/Y_{m+1}\ ,\right.$$

$$\left.\sum_{r_1+1}^{r_2} Y_i/Y_{m+1},\ \ldots,\ \sum_{r_{s-1}+1}^{m} Y_i/Y_{m+1}\right).$$

Since $\left[\sum_{1}^{r_1} Y_i,\ \sum_{r_1+1}^{r_2} Y_i,\ \ldots,\ \sum_{r_{s-1}+1}^{m} Y_i\right]$ has the s–variate Dirichlet distribution $D\left[\sum_{1}^{r_1}\alpha_i,\ \sum_{r_1+1}^{r_2}\alpha_i,\ \ldots,\ \sum_{r_{s-1}+1}^{m}\alpha_i;\ \alpha_{m+1}\right]$ (according to preliminary result 2.3 in section 2). Therefore,

$$(z_1,\ldots,z_s) \sim DL\left[\sum_{1}^{r_1}\alpha_i,\ \sum_{r_1+1}^{r_2}\alpha_i,\ \ldots,\ \sum_{r_{s-1}+1}^{m}\alpha_i;\ \alpha_{m+1}\right]$$

which completes the proof.

Corollary 1. If $(X_1,\ldots,X_m) \sim DL(\alpha_1,\ldots,\alpha_m;\ \alpha_{m+1})$, then

$$-\log \sum_{1}^{m} e^{-x_i} \sim DL(\sum_{1}^{m} \alpha_i;\ \alpha_{m+1}) .$$

Theorem 4.9. If $(X_1,\ldots,X_m) \sim DL(\alpha_1,\ldots,\alpha_m;\ \alpha_{m+1})$, then moment generating function is given by

$$M(t_1,\ldots,t_m) = \frac{\Gamma(\alpha_{m+1} + \sum_{1}^{m} t_i)}{\Gamma(\alpha_{m+1})} \prod_{i=1}^{m} \frac{\Gamma(\alpha_i - t_i)}{\Gamma(\alpha_i)}$$

as a sequel

$$E(X_i) = \frac{\Gamma'(\alpha_{m+1})}{\Gamma(\alpha_{m+1})} - \frac{\Gamma'(\alpha_i)}{\Gamma(\alpha_i)}$$

$$Var(X_i) = \frac{\Gamma''(\alpha_{m+1})}{\Gamma(\alpha_{m+1})} + \frac{\Gamma''(\alpha_i)}{\Gamma(\alpha_i)} - \left[\frac{\Gamma'(\alpha_{m+1})}{\Gamma(\alpha_{m+1})}\right]^2 - \left[\frac{\Gamma'(\alpha_i)}{\Gamma(\alpha_i)}\right]^2$$

$$cov(X_i,X_j) = \frac{\Gamma''(\alpha_{m+1})}{\Gamma(\alpha_{m+1})} - \left[\frac{\Gamma'(\alpha_{m+1})}{\Gamma(\alpha_{m+1})}\right]^2 \quad \text{for } i \neq j$$

where $\Gamma'(\alpha)$ and $\Gamma''(\alpha)$ are the first and the second derivatives of the gamma function $\Gamma(x)$ evaluated at the point $x = \alpha$.

Proof. This follows by applying definition and properties of the moment generating function. The details are omitted.

Theorem 4.10. If the prior distribution of $(X_1,\ldots,X_m)$ is DL–distribution $DL(\alpha_1,\ldots,\alpha_m;\ \alpha_{m+1})$ and if

$$P(X = j \mid X_1,\ldots,X_m) = e^{-x_j} / (1 + \sum_{1}^{m} e^{-x_i}) \text{ a.s. for } j = 1,\ldots,m$$

$$P(X = m+1 \mid X_1,\ldots,X_m) = 1 / (1 + \sum_{1}^{m} e^{-x_i}) \text{ a.s.}$$

then the posterior distribution of $(X_1,\ldots,X_m)$ given $X = j$ is $DL(\alpha_1^{(j)}, \ldots, \alpha_m^{(j)}; \alpha_{m+1}^{(j)})$, where

$$\alpha_i^{(j)} = \alpha_i \quad \text{if } i \neq j$$
$$= \alpha_{j+1} \quad \text{if } i = j \quad \text{for } i, j=1,\ldots,m$$

and

$$\alpha_i^{(m+1)} = \alpha_i \quad \text{for } i=1,\ldots,m$$

$$\alpha_{m+1}^{(m+1)} = \alpha_{m+1} + 1.$$

Proof. According to the condition of theorem, we obtain

$$\int_{\substack{x_i \\ i=1,\ldots,m}} f(j,x_1,\ldots,x_m)\, \pi(x_1,\ldots,x_m) \prod_1^m d\,x_i$$

$$= \int_{\substack{x_i \\ i=1,\ldots,m}} \frac{\Gamma(\alpha)}{\prod_1^{m+1} \Gamma(\alpha_i)} \frac{e^{-x_j}}{(1 + \sum_1^m e^{-x_i})} \frac{e^{-\sum_1^m \alpha_j x_j}}{(1 + \sum_1^m e^{-x_i})^{\alpha}} \prod_1^m d\,x_i$$

$$= \frac{\Gamma(\alpha)}{\prod_1^{m+1} \Gamma(\alpha_i)} \frac{\prod_{i \neq j}^{m+1} \Gamma(\alpha_i)\, \Gamma(\alpha_j+1)}{\Gamma(\alpha+1)} \qquad \text{for } j = 1,\ldots,m$$

where $\alpha = \sum_1^{m+1} \alpha_i$, $\pi(x_1,\ldots,x_m)$ denotes the joint density function of $(X_1,\ldots,X_m)$ ~ $D(\alpha_1,\ldots,\alpha_m; \alpha_{m+1})$ and $f(j,x_1,\ldots,x_m) = P(X=j|x_1,\ldots,x_m)$. Therefore, we obtain

$$\pi(x_1,\dots,x_m|x=j) = \frac{f(j,x_1,\dots,x_m)\pi(x_1,\dots,x_m)}{\int_{x_i,\ 1\le i\le m} f(j,x_1,\dots,x_m)\,\pi(x_1,\dots,x_m)\prod_1^m d\,x_i}$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha_j+1)\prod_{i\neq j}^{m+1}\Gamma(\alpha_i)} \frac{e^{-(\alpha_j+1)x_j}\, e^{-\sum_{i\neq j}^{m}\alpha_i x_i}}{(1+\sum_1^m e^{-x_i})^{\alpha+1}}$$

that is, the posterior distribution of $(X_1,\dots,X_m)$ given $X = j$ is $DL(\alpha_1,\dots,\alpha_{j-1}, \alpha_j+1, \alpha_{j+1},\dots, \alpha_m; \alpha_{m+1})$ for $j=1,\dots,m$.

Using the same way, we may obtain result when $j = m+1$.

5. Sufficient Conditions of DL–distribution.

Fabius (1972), (1973) have studied the relations between Dirichlet distribution and "neutrality". In particular, Fabius (1973) have found two equivalent conditions of Dirichlet distribution. In this paper, since DL–distribution is defined by using Dirichlet distribution, naturally, we may think what is equivalent conditions of DL–distribution. Unfortunately, so far we can not find equivalent conditions. We have found sufficient conditions of DL–distribution. First we define some concepts.

<u>Definition 5.1</u>. A random vector $\underset{\sim}{x} = (X_1,\dots,X_m)$ is called $(CM)_i$–exponential neutral for a given $i \in \{1,\dots,m\}$ iff, for any integers $r_j \geq 0$, $j \neq i$, there is a constant c such that

$$E\left[\prod_{j\neq i}^{m} e^{-r_j X_j} \mid X_i\right] = c(1+e^{-X_i})^{\sum_{j\neq i}^{m} r_j} \qquad \text{a.s.}$$

<u>Definition 5.2</u>. A random vector $\underset{\sim}{x} = (X_1,\dots,X_m)$ is called $(DR)_i$–exponential

neutral for a given $i \in \{1,\dots,m\}$, for any integers $r \geq 0$, there is a constant c such that

$$E(e^{-rX_i} \mid X_j, j \neq i) = c(1 + \sum_{j\neq i}^{m} e^{-X_j})^r, \quad \text{a.s.}$$

<u>Proposition</u>. If $\underset{\sim}{x}$ is nondegenerate, consider the following statements:

(i) $\underset{\sim}{x}$ is $(CM)_i$–exponential neutral for all i;

(ii) $\underset{\sim}{x}$ is $(DR)_i$–exponential neutral for all i;

(iii) The distribution of $\underset{\sim}{x}$ is a DL–distribution or a limit of DL–distribution,

Then we have assertion (ii) ⇒ (i) ⇒ (iii).

<u>Proof</u>. The assertion follows from Lemma 5.2 and Lemma 5.3 below.

To simplify the notation we write

$$e^{-y_{(i)}} = \sum_{j\neq i} e^{-y_j}, \quad e^{-y_I} = \sum_{j\notin I} e^{-y_j}$$

for any

$i \in \{1,\dots,m\}, \quad I \subset \{1,\dots,m\}$

<u>Lemma 5.1</u>. (ii) implies, for any proper subset I of $\{1,\dots,m\}$, any $i \in I$, and any integer $r \geq 0$, the existence of constant c, such that

$$E(e^{-rX_i} \mid X_j,\ j \notin I) = c(1 + e^{-X_I})^r, \quad \text{a.s.}$$

<u>Proof</u>. We proceed by induction with respect to both $|I|$, i.e., the number of elements of I, and r. Let $H_{k,s}$ be the induction hypothesis that the assertion holds for any set I with $|I| \leq k$ and any $r \leq s$. Note that $H_{1,r}$ and $H_{k,0}$ are trivially true for any $r \geq 0$ and $k \in \{1,\dots,m-1\}$. Thus we only need to show that $H_{k,r+1}$ and $H_{k+1,r}$ together imply $H_{k+1,r+1}$ for any $r \geq 0$, $k \in \{1,\dots,m-2\}$. To do this, we must fix a set I with $|I| = k+1$ and $i \in I$. However, without loss of generality we may set $i=1$, $I = \{1,\dots,k+1\}$. Because of $H_{k,r+1}$ we know there is a constant c_1 such that

$$(5.1)\qquad E(e^{-(r+1)X_1} \mid X_j, j \notin I) = E(E(e^{-(r+1)X_1} \mid X_j, j > k) \mid X_j,\ j \notin I)$$

$$= c_1\ E((1+e^{-X_I}+e^{-X_{k+1}})^{r+1} \mid X_j, j \notin I), \qquad \text{a.s.}$$

Moreover, taking expectations, we see that $0 < c_1 < 1$ or $c_1 > 1$.

Expanding the $(1+e^{-X_I}+e^{-X_{k+1}})^{r+1}$ in (5.1) and using $H_{k+1,r'}$, we obtain

$$(5.2)\qquad E(e^{-(r+1)X_1} \mid X_j,\ j \notin I) = c_1\ E(e^{-(r+1)X_{k+1}} \mid X_j, j \notin I) + c_2(1+e^{-X_I})^{r+1}, \qquad \text{a.s.}$$

where c_2 is a constant. In the same way we can show the existence of constant c_1' and c_2' with $0 < c_1' < 1$ or $c_1' > 1$ such that

$$(5.3)\qquad E(e^{-(r+1)X_{k+1}} \mid X_j, j \notin I) = c_1'\ E(e^{-(r+1)X_1} \mid X_j, j \notin I) + c_2'(1+e^{-X_I})^{r+1}, \qquad \text{a.s.}$$

Substitution of (5.3) and (5.2) yields

$$E(e^{-(r+1)X_1} \mid X_j, j \notin I) = \frac{c_1c_2'+c_2}{1-c_1c_1'}(1+e^{-X_I})^{r+1}, \qquad \text{a.s.}$$

and the lemma is proved.

<u>Lemma 5.2</u>. (ii) implies, for any proper subset I of $\{1,\dots,m\}$ and any integers $r_i \geq 0$, $i \in I$, the existence of a constant c, such that

$$E(\prod_{i\in I} e^{-r_iX_i} \mid X_j,\ j \notin I) = c(1+e^{-X_I})^{\sum_{i\in I} r_i}, \qquad \text{a.s.}$$

Thus in particular (ii) implies (i).

<u>Proof</u>. We again use induction on $|I|$, the assertion being trivially true for sets I

with $|I| = 1$. Hence we start out from the assumption that the assertion holds for all sets I with $|I| \leq k$ for a given $k \leq m-2$, and we fix a set I with $|I| = k+1$. As before we put $I = \{1,...,k+1\}$ without loss of generality. Our assumption guarantees the existence, for any integers $r_i \geq 0$, $i \in I$, of a constant c, such that

$$E(\prod_{i\in I} e^{-r_iX_i} \mid X_j, j \notin I) = E(e^{-r_{k+1}X_{k+1}} E(\prod_1^k e^{-r_iX_i} \mid X_j, j > k) \mid X_j, j \notin I)$$

$$= c\, E(e^{-r_{k+1}X_{k+1}} (1+e^{-X_I}+e^{-X_I}+e^{-X_{k+1}})^{\sum_1^k r_i} \mid X_j, j \notin I) \quad \text{a.s.}$$

Expanding $(1+e^{-X_I}+e^{-X_{k+1}})^{\sum_1^k r_i}$ and applying Lemma 5.1 we obtain the desired result.

Lemma 5.3. (i) implies (iii).

Proof. For arbitrary distinct $i, j \in \{1,...,m\}$ and any integers $r, s \geq 0$, the $(CM)_i$ and $(CM)_j$–exponential neutrality implies

$$(5.4) \qquad \frac{E(e^{-rX_i}e^{-sX_j})}{E(e^{-rX_i})E(e^{-sX_j})} = \frac{E(e^{-rX_i}(1+e^{-X_i})^s)}{E(e^{-rX_i})E(1+e^{-X_i})^s} = \frac{E(e^{-sX_j}(1+e^{-X_j})^r)}{E(e^{-sX_j})E(1+e^{-X_j})^r}$$

In particular it follows that $q = E(e^{-X_i}(1+e^{-X_i}))/(E(e^{-X_i})E(1+e^{-X_i}))$ does not depend on i. If either $q = 1$ or $q = \infty$ one easily verifies that the distribution of x is a limit of DL–distribution. In all other cases, $q > 1$, we then define $\alpha, \alpha_1,...,\alpha_m > 0$, put $q = \alpha/(\alpha-1)$ $\alpha \geq r$, $\alpha_i = \alpha\, E(e^{-X_i})$ for $i=1,...,m$. It now turns out that the distribution of $\underset{\sim}{x}$ is the $DL(\alpha_1,...,\alpha_m; \alpha_{m+1})$, where $\alpha_{m+1} \hat{=} \alpha+1$. We first show that all mixed moments can be expressed in terms of marginal moments, or, more precisely, that for any i, any mixed moment of $e^{-X_1},...,e^{-X_i}$ can be expressed in terms of marginal moments of these same random variables. This follows by induction on i. It is trivially true for i=1, and for any i $\leq$ m–1 we have

$$E \prod_{j=1}^{i+1} e^{-r_j X_j} = \frac{E \prod_{1}^{i} e^{-r_j X_j}}{E(1+e^{-X_{i+1}})^{\sum_1^i r_j}} \; E(e^{-r_{i+1} X_{i+1}}(1+e^{-X_{i+1}})^{\sum_1^i r_j})$$

by the $(CM)_{i+1}$–exponential neutrality.

It remains to be shown that the marginal moments of the e^{-X_i} have the right value. This is in fact true for the first moments by the definition of the α_i. Supposing it to be true for all moments of order not exceeding a given r $\geq$ 1, we may use (5.4) to obtain

$$\text{(5.5)} \qquad E(e^{-rX_i}(1+e^{-X_i})) = \frac{E(e^{-rX_i})E(1+e^{-X_i})}{E(e^{-X_j})E(1+e^{-X_j})^r} \; E(e^{-X_j}(1+e^{-X_j})^r)$$

and

$$\text{(5.6)} \qquad E(e^{-(r-1)X_i}(1+e^{-X_i})^2) = \frac{E(e^{-(r-1)X_i})E(1+e^{-X_i})^2}{E(e^{-2X_j})E(1+e^{-X_j})^{r-1}} \; E(e^{-2X_j}(1+e^{-X_j})^{r-1})$$

for arbitrary distinct i and j. Adding (5.5) and (5.6) we obtain an expression for $E(e^{-(r-1)X_i}(1+e^{-X_i}))$ involving $E(e^{-(r+1)X_j})$ and moments of order not exceeding r. Up to its sign the coefficient of $E(e^{-(r+1)X_j})$ in this expression is given by

$$\frac{E(e^{-r-1)X_i})E(1+e^{-X_i})^2}{E(e^{-2X_j})E(1+e^{-X_j})^2} - \frac{E(e^{-rX_i})E(1+e^{-X_i})}{E(e^{-X_j})E(1+e^{-X_j})^r}$$

$$= \frac{\alpha_i(\alpha_i+1)\ldots(\alpha_i+r-2)(a+\alpha_i)}{\alpha_j(\alpha+\alpha_j)\ldots(\alpha+\alpha_j-r+2)} \left\{\frac{\alpha+\alpha_i-1}{\alpha_j+1} - \frac{\alpha_i+r-1}{\alpha+\alpha_j-r+1}\right\} > 0$$

and hence does not vanish. Thus we can solve for $E(e^{-(r+1)X_j})$, expression it in terms of moments of lower order. Without further computation we may conclude that all moments of order r+1 coincide with those of

$$(Y_1/Y_{m+1},\dots,Y_m/Y_{m+1})$$

where

$$(Y_1,\dots,Y_m) \sim D(\alpha_1,\dots,\alpha_m;\ \alpha+1),\ Y_{m+1} \hat{=} 1 - \sum_1^m Y_i.$$

Therefore

$$(e^{-X_1},\dots,e^{-X_m}) \overset{d}{=} (Y_1/Y_{m+1},\dots,Y_m/Y_{m+1}),$$

that is

$$(X_1,\dots,X_m) \sim DL(\alpha_1,\dots,\alpha_m;\ \alpha+1)$$

and thus the proof is completed.

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