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Change-in-ratio Estimators for Populations With More Than Two Subclasses

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ABSTRACT

Change-in-ratio methods have been developed to estimate the size of populations with 2 or 3 population subclasses. Most of these methods require the often unreasonable assumption of equal sampling probabilities for individuals in all subclasses. This paper presents new models which provide explicit estimates of population size for populations with 3 or more subclasses. The new models are based on the weaker assumption that sampling probabilities are equal for only 2 of the subclasses and that the ratios of the sampling probabilities remain constant over time. Extensions of these models which require an additional sample but further relax the assumption of equal sampling probabilities are discussed. Likelihood ratio tests can be used to select the most appropriate model for a particular set of data. Emphasis is on the 2 sample, 3 subclass models for which Monte Carlo simulation results and an illustrative example are presented.

Key words: Change-in-ratio; Population estimation; Unequal sampling probabilities

1. Introduction

Change-in-ratio (CIR) methods are attractive for estimating fish and wildlife population sizes because, relative to other methods, the required data are often easier and less expensive to obtain. Seber (1982) has provided the most recent review of these methods.

The essential requirements of CIR methods are that the population be partitioned into subclasses and that the subclass proportions change due to disproportionate removals from (or additions to) the subclasses. CIR methods have been developed for populations partitioned into 2 (Kelker 1940, Chapman 1955, Pollock et. al. 1985) or 3 (Otis 1980) subclasses. The change in subclass proportions provides the information required to estimate population size. Subclass proportions are estimated from samples taken before and after each removal. Most CIR methods require that the probability of being sampled be the same for individuals of all subclasses (Kelker 1940, Chapman 1955, Otis 1980). Pollock et. al. (1985) presented an extension of the methodology for a population partitioned into 2 subclasses which relaxed this assumption, requiring only that the ratio of the sampling probabilities remains constant over time.

This paper presents analogous extensions for populations partitioned into 3 or more subclasses. One set of extensions uses 2 samples and requires that the sampling probabilities be equal for only 2 of the subclasses. The other set of extensions uses 3 samples and allows a different sampling probability for each of the subclasses. All of the extensions require that the ratios of the sampling probabilities remain constant over time. Section 2 provides a brief description of the Otis (1980) 2 sample, 3 subclass model along with an alternative method for estimating its parameters. This model is generalized to allow the sampling probability for 1 of the subclasses to differ from that for the others in Section 3.

Sections 4 and 5 contain results of Monte Carlo simulations used to compare the 2 sample, 3 subclass estimators and a hypothetical example illustrating estimation and model selection under these models. Further generalizations of the model that allow for more than 3 subclasses and make use of 3 samples are given in Section 6. Section 7 concludes with some general remarks.

2. Otis Method

In order to allow generalization of the method, the notation of Otis (1980) will be altered as follows. Let

X_{ij} = the number of type i individuals in the population at time j, i=1,2,3, j=1,2.

$N_j = \sum_{i=1}^{3} X_{ij}$ = the total population size at time j, j=1,2.

x_{ij} = the number of type i individuals in the sample at time j, i=1,2,3, j=1,2.

$n_j = \sum_{i=1}^{3} x_{ij}$ = the total sample size at time j, j=1,2.

R_i = the number of type i individuals removed from the population between the 2 sample times, i=1,2,3.

$R = \sum_{i=1}^{3} R_i$ = the total number of individuals of all types removed from the population between the 2 sample times.

The population is assumed closed except for the known removals, R_1, R_2, and R_3, and possibly, an unknown mortality that has the same proportional affect on each of the subclasses. The first sample is taken before, and the second sample after, the removals. Otis (1980) modeled each of the sample counts (x_{1j}, x_{2j}), $j=1,2$ as independent trinomial random variables with expectations given by

$$\mathcal{E}(x_{ij}) = \frac{X_{ij}}{N_j} .$$

In more general terms, the expected value of the sample count of individuals in a subclass was assumed to be proportional to the number of individuals in that population subclass. The coefficient of proportionality $\gamma_{ij} = 1/N_j$ was assumed to be the same for all subclasses at any given time. This is the assumption of equal sampling probabilities for all subclasses. Under this model, the log likelihood is proportional to

$$\sum_{j=1}^{2}\sum_{i=1}^{3} x_{ij} \ln\left(\frac{X_{ij}}{N_j}\right) , \quad (1)$$

where $X_{i2} = X_{i1} - R_i$ and $N_2 = N_1 - R$. The maximum likelihood estimates cannot be expressed in closed form and Otis (1980) gives the estimates in terms of an auxiliary parameter which must be found numerically as one of the roots of a 5'th degree polynomial.

An alternative approach to obtaining the maximum likelihood estimates uses the method of iteratively reweighted nonlinear least squares which is available in many statistical software packages (Jennrich and Moore 1975). This approach is based on the equivalence of the maximum likelihood estimates from (1) and the generalized least squares estimates for the nonlinear model

$$x_{ij} = \mathcal{E}(x_{ij}) + e_{ij}, \quad i=1,2, \quad j=1,2 ,$$

where

$$\mathcal{E}(x_{1j}) = n_j(X_{1j}/N_j)$$

$$\mathcal{E}(x_{2j}) = (n_j - x_{1j})\left(\frac{X_{2j}/N_j}{1-(X_{1j}/N_j)}\right)$$

$$\mathrm{Var}(e_{1j}) = n_j\left(\frac{X_{1j}}{N_j}\right)\left[1-\left(\frac{X_{1j}}{N_j}\right)\right]$$

$$\mathrm{Var}(e_{2j}) = (n_j - x_{1j})\left[\frac{X_{2j}/N_j}{1-(X_{1j}/N_j)}\right]\left[1-\left(\frac{X_{2j}/N_j}{1-(X_{1j}/N_j)}\right)\right]$$

$$\mathrm{Cov}(e_{ij}, e_{kl}) = 0 \quad \text{for } i \neq k \text{ or } j \neq l \ .$$

The trinomial distribution of the sample counts at each time has been re-expressed as a pair of conditionally independent binomial distributions so that there are 4 conditionally independent univariate observations for the regression. The regression must be performed iteratively, using parameter estimates from the previous step to obtain new estimates of $\mathrm{Var}(e_{ij})$ at each step. The code for implementing this procedure with PROC NLIN (SAS 1985) is given in Appendix 2.

The maximum likelihood estimates must satisfy

$$0 \leq \hat{X}_{11}/\hat{N}_1 \leq 1 \qquad\qquad 0 \leq \hat{X}_{21}/\hat{N}_1 \leq 1$$

$$0 \leq (\hat{X}_{11}-R_1)/(\hat{N}_1-R) \leq 1 \qquad\qquad 0 \leq (\hat{X}_{21}-R_2)/(\hat{N}_1-R) \leq 1,$$

but it may be that $\hat{X}_{i1} < R_i$ or even $\hat{X}_{i1} < 0$. Clearly, if the maximum likelihood estimate of an initial subclass size is less than the removal from that subclass, then it is biologically meaningless and the method must be considered to have failed. Likewise, though there is no upper limit on the maximum likelihood estimates of subclass size, the method must be considered to have failed if the estimates are unreasonably large. All of the CIR methods are susceptible to these types of failure. They can occur when the direction of change for the sample proportions is unrepresentative of that for the true population proportions, an event which always has some positive probability regardless of sample size or the true parameter values.

The PROC NLIN (SAS 1985) estimation procedure should be repeated using various starting values for the parameters to insure that the final estimates correspond to a global maximum. Some of these starting values should be unreasonably large and some should be less than the removals since the maximum likelihood estimates will not necessarily occur in the biologically meaningful portion of the parameter space.

3. New Method

Often it will not be reasonable to assume that the sampling probabilities are the same for all 3 subclasses. A new set of estimators which relaxes this assumption can be derived based on moment considerations. The population is assumed closed except for known removals. At each time, it is assumed that the expected value of the sample count of individuals in a subclass is proportional to the number of individuals in that population subclass. That is,

$$\mathcal{E}(x_{ij}) = \gamma_{ij} X_{ij} \ . \tag{2}$$

It is further assumed that at any given time, the probabilities of sampling individuals from subclasses 1 and 2 are the same and that the probability of sampling individuals from subclass 3 is λ_3 times the probability of sampling individuals from subclasses 1 or 2. These assumptions are equivalent to the constraints

$$\gamma_{1j}=\gamma_{2j}, \quad j=1,2$$
$$\gamma_{31}/\gamma_{11} = \gamma_{32}/\gamma_{12} = \lambda_3 \; . \tag{3}$$

Recalling that $X_{i2}=X_{i1}-R_i$ and setting the sample counts equal to their expectations subject to the constraints (3) gives a system of 6 equations with 6 unknown parameters which can be solved to obtain the following estimators.

$$\hat{X}_{11} = \frac{x_{11}(x_{22}R_1-x_{12}R_2)}{x_{11}x_{22}-x_{12}x_{21}}$$

$$\hat{X}_{21} = \frac{x_{21}\hat{X}_{11}}{x_{11}}$$

$$\hat{\lambda}_3 = \frac{(x_{22}x_{31}-x_{21}x_{32})R_1+(x_{11}x_{32}-x_{12}x_{31})R_2}{(x_{11}x_{22}-x_{12}x_{21})R_3} \tag{4}$$

$$\hat{X}_{31} = \frac{x_{31}\hat{X}_{11}}{\hat{\lambda}_3 x_{11}} \; .$$

The asymptotic variances and covariances of these estimators can be obtained from Taylor series approximations and are given in Appendix 1.

If, in addition to the moment assumptions, it is assumed that the sample counts at each time are distributed as independent trinomials, then the log likelihood is proportional to

$$\sum_{j=1}^{2}\sum_{i=1}^{3} x_{ij} \ln\left[\frac{\gamma_{ij}X_{ij}}{\sum_{i=1}^{3}\gamma_{ij}X_{ij}}\right],$$

subject to the constraints (3). Under this distributional assumption, the new estimators are maximum likelihood. The new estimators will be maximum likelihood under any distribution for which the sample proportions are maximum likelihood estimates of the population proportions. Note that the Otis (1980) model is a special case of this model, obtained by setting $\lambda=1$.

Though it will usually be most convenient to obtain these estimates using the explicit forms (4), they may also be obtained using iteratively reweighted nonlinear least squares as described for the Otis (1980) estimates. This latter approach may be attractive if both sets of estimates are required, because the code for estimation under the 2 models differs only by the placement of a semicolon (see Appendix 2). The PROC NLIN (SAS 1985) implementation of this procedure (Appendix 2) is also a covenient method for obtaining the standard errors and correlations of the estimates.

4. Comparison of Estimators

Sample counts were simulated as

$$(x_{11}, x_{21}) \sim \text{trinomial}\left(n_1, \frac{X_{11}}{X_{11}+X_{21}+\lambda_3X_{31}}, \frac{X_{21}}{X_{11}+X_{21}+\lambda_3X_{31}}\right)$$

$$(x_{12}, x_{22}) \sim \text{trinomial}$$

$$\left(n_2, \frac{X_{11}-R_1}{X_{11}-R_1+X_{21}-R_2+\lambda_3(X_{31}-R_3)}, \frac{X_{21}-R_2}{X_{11}-R_1+X_{21}-R_2+\lambda_3(X_{31}-R_3)}\right).$$

Parameter values were selected to correspond to a range of removal rates, sample sizes and relative sampling probabilities. At least 500 (1000 maximum) replicate sets of counts were generated for each set of parameter values. Estimates of the initial population size were obtained for each set of counts under the new model using the explicit estimators (4) and under the Otis (1980) model using PROC NLIN (SAS 1985).

The true parameter values were used as starting values for obtaining the Otis (1980) estimates. The large number of replicates made it infeasible to repeat the estimation procedure with other starting values. However, our detailed investigation of several likelihood surfaces indicated that the surfaces were smooth and unimodal in a broad region around the maximum. Thus, if the maximum likelihood estimates were within a reasonable proximity of the true parameter values, they should have been found.

Estimates were compared based on the proportion within a given distance of the true value. This criterion is not affected by the possibility that some estimates might correspond to local rather than global maxima (assuming that those estimates and the estimates corresponding to the global maxima are both beyond the given distance from the true values). Also, this criterion is not sensitive to the occasional extreme values produced by CIR methods (method failures, as discussed in Section 2). Statistics such as the mean and standard error are quite sensitive to extreme values and are unstable measures of location and spread for these distributions.

The proportions of the estimates within specified ranges of the true value for various removal structures, sample sizes and relative sampling probabilities are given in Tables 1 and 2. In all cases, the new estimator had a distribution which was skewed to the right with a peak somewhat below the true value. The distribution of the Otis (1980) estimator was also skewed to the right, but the location of its peak was quite variable. When the sampling probability was the same for all subclasses ($\lambda_3=1$), the peak of this distribution was located somewhat below the true value.

When $\lambda_3=1$, both estimators were reasonably well centered on the true value, but the distribution of the Otis (1980) estimator had consistently less spread than that of the new estimator. The proportion of estimates within a given range of the true value increased with increasing sample size for both estimators.

When the sampling probability for subclass 3 was different from that for the other subclasses ($\lambda_3 \neq 1$), the new estimator was centered more closely on the true value than the Otis (1980) estimator in every case except one ($R_1/X_{11}=.2$, $R_2/X_{21}=.4$, $R_3/X_{31}=.2$, $\lambda_3=.5$). For half of these cases, the distributions of the new estimator also had less spread than the distribution of the Otis (1980) estimator. For the new estimator, both the spread of the distribution and the proportion of the estimates within a given range of the true value tended to improve as the sample size increased or as $\mathcal{E}\{(x_{22}x_{31}-x_{21}x_{32})R_1+(x_{11}x_{32}-x_{12}x_{31})R_2\}$ or $\mathcal{E}\{x_{11}x_{22}-x_{12}x_{21}\}$ increased in magnitude.

Since the Otis (1980) model is actually a restriction of the new model obtained by setting $\lambda_3=1$, the ratio of their likelihoods can be used to test whether the assumption that $\lambda_3=1$ is supported by a given set of data. According to standard theory, the asymptotic distribution of -2 ln(likelihood ratio) will be chi square with 1 degree of freedom under the null hypothesis that $\lambda_3=1$. Tables 3 and 4 give the proportion of the simulated datasets for which this null hypothesis was rejected at various significance levels. The rejection rates for $\lambda_3=1$ were in good agreement with the nominal significance levels of the tests. The power of the test against alternatives of $\lambda_3 \neq 1$ was highly dependent on other parameter values, but increased with increasing sample size.

5. Example

Consider a population partitioned into 3 subclasses. A random sample of 500 individuals is taken with replacement. 128 of these individuals are classified as type 1, 119 as type 2 and the remaining 253 as type 3. Following this sample, there is a harvest in which 140 type 1, 280 type 2 and 560 type 3 individuals are removed from the population. A second random sample of 500 individuals is taken with replacement after the harvest. 227 individuals from the second sample are classified as type 1, 167 as type 2 and 106 as type 3.

Estimates of the initial subclass sizes under the Otis (1980) model were obtained with PROC NLIN (SAS 1985) using the code in Appendix 2 and a variety of starting values (Table 5). The maximum likelihood estimates and their standard errors are given in Table 6. Estimates of the asymptotic standard errors and correlations were obtained from the PROC NLIN (SAS 1985) output. The correlations are given in Table 7.

Estimates of the initial subclass sizes and the ratio of type 1 or 2 to type 3 sampling probabilities under the new model were obtained using the explicit expressions (4) and their covariances were obtained using the expressions in Appendix 1. The estimates and their standard errors are given in Table 6. The estimated correlations are given in Table 8. These estimates and their covariances were also obtained with PROC NLIN (SAS 1985) using the code in Appendix 2 with λ_3 (LAMBDA3) designated as a parameter.

For testing the null hypothesis that $\lambda_3=1$, we have $-2\ \ln(\text{likelihood ratio}) = 8.7$, so that this hypothesis is rejected ($p<.01$). This indicates it is not appropriate to assume that the sampling probability for type 3 individuals is the same as for other individuals. Estimates from the new model should be used in this case.

6. Generalizations

6.1 More than 3 subclasses

The new method can be generalized to provide explicit estimators for populations with more than 3 subclasses. This generalization may be particularly useful in the context of fish populations, for example, where the subclasses may consist of size classes, of which there could well be more than 3.

The extension retains the assumptions (2) and (3) and assumes with respect to any additional subclasses that

$$\gamma_{i1}/\gamma_{11} = \gamma_{i2}/\gamma_{12} = \lambda_i, \quad i=4,\cdots,t \ . \tag{5}$$

These assumptions are analogous to the assumption concerning the relative probability of sampling individuals from subclass 3; the probability of sampling an individual from subclass i, $i>2$ is λ_i times the probability of sampling an individual from subclass 1 or 2.

Setting the sample counts equal to their expectations subject to the constraints (3) and (5) and solving gives the estimators for X_{11} and X_{21} given in (4) and

$$\hat{\lambda}_i = \frac{(x_{22}x_{i1}-x_{21}x_{i2})R_1+(x_{11}x_{i2}-x_{12}x_{i1})R_2}{(x_{11}x_{22}-x_{12}x_{21})R_i}, \qquad i=3,\cdots,t$$

$$\hat{X}_{i1} = \frac{x_{i1}\hat{X}_{11}}{\hat{\lambda}_i x_{11}}, \qquad i=3,\cdots,t \ . \tag{6}$$

Under the additional assumption that the sample counts $(x_{1j},x_{2j},\cdots,x_{t-1,j})$, $j=1,2$ at each time are distributed as independent multinomials, the log likelihood is proportional to

$$\sum_{j=1}^{2}\sum_{i=1}^{t} x_{ij} \ln\left[\frac{\gamma_{ij}X_{ij}}{\sum_{i=1}^{t}\gamma_{ij}X_{ij}}\right],$$

subject to constraints (3) and (5). Again, the explicit estimators under the new model are maximum likelihood.

Reduced models may be formed by setting any of the $\lambda_i=1$ and the maximum likelihood estimators under any of these reduced models may be obtained with iteratively reweighted nonlinear least squares. Likelihood ratio tests may be used to select the most appropriate model.

6.2 Two removal periods and 3 samples

In some cases, it may be possible to follow the second sample with another period of removals and then obtain a third sample. Chapman (1955) and Pollock et. al. (1985) have presented extensions of CIR methodology for populations with 2 subclasses that make use of this type of additional information. For a population with 3 or more subclasses, the addition of another removal period followed by a third sample allows a relaxation of the requirement that at least 2 subclasses have equal sampling probabilities.

In general, if the sample counts $(x_{1j},x_{2j},\cdots,x_{t-1,j})$, j=1,2,3 are distributed as independent multinomials with expectations proportional to the sizes of the population subclasses, then the log likelihood will be proportional to

$$\sum_{j=1}^{3}\sum_{i=1}^{t} x_{ij} \ln\left[\frac{\gamma_{ij}X_{ij}}{\sum_{i=1}^{t}\gamma_{ij}X_{ij}}\right]. \tag{7}$$

Maximum likelihood estimates of the initial subclass sizes can be obtained if at least t constraints are imposed on the γ_{ij}, $i=1,\cdots,t$, $j=1,2,3$. Allowing the sampling probabilities to be different for each subclass, but requiring that the ratios of these probabilities remain constant over time is equivalent to the 2t−1 constraints

$$\frac{\gamma_{i1}}{\gamma_{11}} = \frac{\gamma_{i2}}{\gamma_{12}} = \lambda_i, \text{ and } \frac{\gamma_{i1}}{\gamma_{11}} = \frac{\gamma_{i3}}{\gamma_{13}} = \lambda_i, \quad i=2,\cdots,t. \tag{8}$$

Maximum likelihood estimates of X_{i1}, $i=1,\cdots,t$ and λ_i, $i=2,\cdots,t$ based on the likelihood (7) subject to the constraints (8) can be obtained by the method of iteratively reweighted nonlinear least squares. In effect, the third sample provides the additional information required to estimate λ_2. Reduced models may be formed by setting any of the $\lambda_i=1$ and likelihood ratio tests may be used to select the most appropriate model.

7. Discussion

Populations may be partitioned into subclasses on the basis of such factors as sex, age and size. There are many situations where there could be interest in more than 2 subclasses and it would be desirable to estimate the size of each. CIR methods require that the population be partitioned into at least enough subclasses so that the sampling probability is the same for individuals within any given subclass. Otis (1980) suggested that the requirement to consider 3 subclasses might arise when interest was primarily in male and female subclasses, but juveniles which could not be accurately sexed were present. The ability to consider more than 2 subclases with unequal sampling probabilities also opens the possibility of applying the method to whole communities where each species might be considered as a subclass.

The methodology presented in this paper provides a general approach for estimating the size of each subclass in a population partitioned into 3 or more subclasses based on CIR data. The nested character of the models allows sequential use of likelihood ratio tests to assess the validity of assuming that individuals in additional subclasses have the same sampling probabilities. The advantage of using a reduced model which assumes more of the sampling probabilities to be equal is that there are fewer parameters to estimate and if the assumption is valid, the standard errors will be smaller. However, if the assumption of equal sampling probabilities is not valid, these estimates will be biased and should be rejected in favor of estimates obtained under the model which relaxes this assumption.

The estimators have explicit forms for the 2 sample models with individuals in only 2 of the subclasses assumed to have equal sampling probabilities. These estimators are quite general, in that they are based only on moment assumptions and do not require the assumption of a specific distribution for the sample counts. The estimators for the 3 sample models and the other 2 sample models (including the Otis (1980) model) do not have explicit forms and are found by numerically maximizing a likelihood which must be obtained by assuming a specific distribution for the sample counts.

These models can easily be extended to use information from more than 3 samples, but the closed population requirement puts a practical limit on the number of sampling periods. Other CIR estimators can be obtained by imposing different types of constraints on the γ_{ij}, $i=1,\cdots,t$, $j=1,\cdots,s$. The models presented here appear to be the most generally applicable, but other types of constraints may be indicated in specific applications.

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Table 1. Distribution of $\hat{N}_1$ with respect to the true value for the Otis (1980) estimator and the new estimator over a range of removal structures and relative sampling probabilities ($X_{11}=X_{21}=X_{31}=700$, $n_1=n_2=500$).

			$\lambda_3=0.5$		$\lambda_3=1.0$		$\lambda_3=2.0$	
R_1/X_{11}	R_2/X_{21}	R_3/X_{31}	Otis	New	Otis	New	Otis	New
.2	.4	.2	16[a]	15	18	13	11	12
			39[b]	37	47	34	22	32
			71[c]	67	79	67	38	63
		.4	6	20	21	20	7	18
			14	50	51	49	19	43
			28	77	78	76	30	73
		.8	0	27	65	26	0	20
			0	63	97	63	5	53
			15	82	100	82	100	78
.4	.8	.2	11	26	68	27	0	20
			76	61	96	59	16	53
			100	76	100	74	69	74
		.4	43	45	57	43	1	43
			94	83	93	81	9	77
			100	95	99	92	51	92
		.8	3	75	78	76	48	64
			26	99	99	98	85	96
			81	100	100	100	97	100

[a] Percent of estimates within 10% of true value

[b] Percent of estimates within 25% of true value

[c] Percent of estimates within 50% of true value

Table 2. Distribution of $\hat{N}_1$ with respect to the true value for the Otis (1980) estimator and the new estimator over a range of removal structures and relative sampling probabilities ($X_{11}=X_{21}=X_{31}=700$, $n_1=n_2=1000$).

			$\lambda_3=0.5$		$\lambda_3=1.0$		$\lambda_3=2.0$	
R_1/X_{11}	R_2/X_{21}	R_3/X_{31}	Otis	New	Otis	New	Otis	New
.2	.4	.2	26[a]	21	26	17	4	14
			54[b]	49	61	46	11	37
			82[c]	76	85	75	29	71
		.4	2	29	29	24	4	25
			9	65	67	66	11	59
			24	85	88	87	21	81
		.8	0	40	81	36	0	32
			0	75	99	72	2	72
			5	90	100	88	100	88
.4	.8	.2	5	32	84	33	0	28
			84	68	100	69	6	65
			100	80	100	82	80	79
		.4	47	60	76	60	0	49
			98	92	99	93	2	88
			100	98	100	98	53	97
		.8	1	92	91	89	62	80
			22	100	100	100	92	99
			89	100	100	100	99	100

a Percent of estimates within 10% of true value

b Percent of estimates within 25% of true value

c Percent of estimates within 50% of true value

Table 3. Percent of replicates for which the likelihood ratio test rejected the null hypothesis that $\lambda_3=1$ at various significance levels (α) for a range of removal structures and true values of λ_3 ($X_{11}=X_{21}=X_{31}=700$, $n_1=n_2=500$).

				λ_3		
R_1/X_{11}	R_2/X_{21}	R_3/X_{31}	α	0.5	1.0	2.0
.2	.4	.2	.01	11	1	3
			.05	25	5	9
			.10	37	10	15
		.4	.01	15	1	25
			.05	31	5	48
			.10	43	12	60
		.8	.01	8	1	41
			.05	21	4	65
			.10	31	9	76
.4	.8	.2	.01	8	1	2
			.05	22	4	8
			.10	34	7	13
		.4	.01	64	1	12
			.05	83	7	25
			.10	90	11	37
		.8	.01	95	1	100
			.05	98	5	100
			.10	99	10	100

Table 4. Percent of replicates for which the likelihood ratio test rejected the null hypothesis that $\lambda_3=1$ at various significance levels (α) for a range of removal structures and true values of λ_3 ($X_{11}=X_{21}=X_{31}=700$, $n_1=n_2=1000$).

				λ_3		
R_1/X_{11}	R_2/X_{21}	R_3/X_{31}	α	0.5	1.0	2.0
.2	.4	.2	.01	26	1	4
			.05	50	6	14
			.10	59	11	21
		.4	.01	28	2	50
			.05	52	5	73
			.10	65	10	84
		.8	.01	17	1	74
			.05	37	5	90
			.10	48	8	94
.4	.8	.2	.01	22	2	2
			.05	44	6	10
			.10	56	10	20
		.4	.01	95	1	26
			.05	98	4	46
			.10	99	8	57
		.8	.01	100	1	100
			.05	100	4	100
			.10	100	9	100

Table 5. Parameter estimates obtained under the Otis (1980) model using PROC NLIN (SAS 1985) with a variety of starting values.

Starting value			Parameter estimates			
X_{11}	X_{21}	X_{31}	X_{11}	X_{21}	X_{31}	L^{a}
700	700	700	317	401	642	−1049
20000	20000	20000	317	401	642	−1049
2100	1400	700	317	401	642	−1049
100	100	100	3.4×10^{12}	2.7×10^{12}	3.4×10^{12}	−1093
−500	−500	−500	1.5×10^{8}	1.2×10^{8}	1.5×10^{8}	−1093

[a] $L = \ln(\text{likelihood}) - \ln\left(\dfrac{n_1!\ n_2!}{x_{11}!\ x_{21}!\ x_{31}!\ x_{12}!\ x_{22}!\ x_{32}!}\right)$

Table 6. Parameter estimates and their standard errors obtained under the Otis (1980) model and under the new model.

	Otis		New	
Parameter	Estimate	Std. Error	Estimate	Std. Error
X_{11}	317	45	912	632
X_{21}	401	33	848	495
X_{31}	642	24	700	43
λ_3	---	---	2.58	1.55

Table 7. Estimated correlations for parameter estimates obtained under the Otis (1980) model.

	X_{11}	X_{21}	X_{31}
X_{11}	1.000	.933	.918
X_{21}		1.000	.909
X_{31}			1.000

Table 8. Estimated correlations for parameter estimates obtained under the new model.

	X_{11}	X_{21}	X_{31}	λ_3
X_{11}	1.000	.995	.816	.980
X_{21}		1.000	.816	.980
X_{31}			1.000	.713
λ_3				1.000

Appendix 1

$$\mathrm{Var}(\hat{X}_{i1}) = \left(\frac{-x_{i2}\hat{X}_{21}}{x_{11}x_{22} - x_{12}x_{21}}\right)^2 \mathrm{Var}(x_{11}) + \left(\frac{x_{i1}\hat{X}_{22}}{x_{11}x_{22} - x_{12}x_{21}}\right)^2 \mathrm{Var}(x_{12})$$

$$+ \left(\frac{x_{i2}\hat{X}_{11}}{x_{11}x_{22} - x_{12}x_{21}}\right)^2 \mathrm{Var}(x_{21}) + \left(\frac{-x_{i1}\hat{X}_{12}}{x_{11}x_{22} - x_{12}x_{21}}\right)^2 \mathrm{Var}(x_{22})$$

$$- 2\left(\frac{x_{i2}^2\hat{X}_{11}\hat{X}_{21}}{(x_{11}x_{22} - x_{12}x_{21})^2}\right) \mathrm{Cov}(x_{11},x_{21}) - 2\left(\frac{x_{i1}^2\hat{X}_{12}\hat{X}_{22}}{(x_{11}x_{22} - x_{12}x_{21})^2}\right) \mathrm{Cov}(x_{12},x_{22}), \quad i=1,2$$

$$\mathrm{Cov}(\hat{X}_{11},\hat{X}_{21}) = \left(\frac{x_{12}x_{22}\hat{X}_{21}^2}{(x_{11}x_{22} - x_{12}x_{21})^2}\right) \mathrm{Var}(x_{11})$$

$$+ \left(\frac{x_{11}x_{21}\hat{X}_{22}^2}{(x_{11}x_{22} - x_{12}x_{21})^2}\right) \mathrm{Var}(x_{12}) + \left(\frac{x_{12}x_{22}\hat{X}_{11}^2}{(x_{11}x_{22} - x_{12}x_{21})^2}\right) \mathrm{Var}(x_{21})$$

$$+ \left(\frac{x_{11}x_{21}\hat{X}_{12}^2}{(x_{11}x_{22} - x_{12}x_{21})^2}\right) \mathrm{Var}(x_{22}) - 2\left(\frac{x_{12}x_{22}\hat{X}_{11}\hat{X}_{21}}{(x_{11}x_{22} - x_{12}x_{21})^2}\right) \mathrm{Cov}(x_{11},x_{21})$$

$$- 2\left(\frac{x_{11}x_{21}\hat{X}_{12}\hat{X}_{22}}{(x_{11}x_{22} - x_{12}x_{21})^2}\right) \mathrm{Cov}(x_{12},x_{22})$$

$$\mathrm{Var}(\hat{X}_{31}) = \left[\frac{x_{32}\,(x_{21}R_1 - x_{11}R_2 - x_{31}R_2)\,\hat{X}_{31}}{x_{31}\left[(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2\right]}\right]^2 \mathrm{Var}(x_{11}) +$$

$$\left[\frac{-R_2}{(x_{22}R_1 - x_{12}R_2)} + \frac{-x_{21}R_1 + x_{11}R_2 + x_{31}R_2}{(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2}\right]^2 \hat{X}_{31}^2\ \mathrm{Var}(x_{12})$$

$$+ \left[\frac{x_{32}\,(x_{21}R_1 + x_{31}R_1 - x_{11}R_2)\,\hat{X}_{31}}{x_{31}\left[(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2\right]}\right]^2 \mathrm{Var}(x_{21}) +$$

$$\left[\frac{R_1}{(x_{22}R_1 - x_{12}R_2)} + \frac{-x_{21}R_1 - x_{31}R_1 + x_{11}R_2}{(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2}\right]^2 \hat{X}_{31}^2\ \mathrm{Var}(x_{22}) +$$

$$2\left(\frac{x_{32}^2\,(x_{21}R_1 - x_{11}R_2 - x_{31}R_2)\,(x_{21}R_1 + x_{31}R_1 - x_{11}R_2)}{x_{31}^2\left[(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2\right]^2}\right) \hat{X}_{31}^2\ \mathrm{Cov}(x_{11},x_{21})$$

$$+ 2\left\{\left\{\frac{-R_2}{(x_{22}R_1 - x_{12}R_2)} + \frac{-x_{21}R_1 + x_{11}R_2 + x_{31}R_2}{(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2}\right\} \times\right.$$

$$\left.\left\{\frac{R_1}{(x_{22}R_1 - x_{12}R_2)} + \frac{-x_{21}R_1 - x_{31}R_1 + x_{11}R_2}{(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2}\right\}\right\} \hat{X}_{31}^2\ \mathrm{Cov}(x_{12},x_{22})$$

$$\mathrm{Var}\ (\hat{\lambda}_3) = \left[\frac{-x_{22}R_1 + x_{12}R_2 + x_{32}R_2 - x_{22}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\ R_3}\right]^2 \mathrm{Var}(x_{11})$$

$$+ \left[\frac{x_{21}R_1 - x_{11}R_2 - x_{31}R_2 + x_{21}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\ R_3}\right]^2 \mathrm{Var}(x_{12})$$

$$+ \left[\frac{-x_{22}R_1 - x_{32}R_1 + x_{12}R_2 + x_{12}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\ R_3}\right]^2 \mathrm{Var}(x_{21})$$

$$+ \left[\frac{x_{21}R_1 + x_{31}R_1 - x_{11}R_2 - x_{11}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\ R_3}\right]^2 \mathrm{Var}(x_{22})$$

$$+ \left[\frac{-x_{22}R_1 + x_{12}R_2 + x_{32}R_2 - x_{22}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\ R_3} \times \frac{-x_{22}R_1 - x_{32}R_1 + x_{12}R_2 + x_{12}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\ R_3}\right] \mathrm{Cov}(x_{11},x_{21})$$

$$+ \left[\frac{x_{21}R_1 - x_{11}R_2 - x_{31}R_2 + x_{21}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\ R_3} \times \frac{x_{21}R_1 + x_{31}R_1 - x_{11}R_2 - x_{11}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\ R_3}\right] \mathrm{Cov}(x_{12},x_{22})$$

$$
\begin{aligned}
\mathrm{Cov}(\hat{X}_{11},\hat{X}_{31}) = & \left\{ \frac{-x_{12}\hat{X}_{21}}{x_{11}x_{22} - x_{12}x_{21}} \times \right. \\
& \left. \frac{x_{32}\,(x_{21}R_1 - x_{11}R_2 - x_{31}R_2)\,\hat{X}_{31}}{x_{31}\left[(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2\right]} \right\} \mathrm{Var}(x_{11}) \\
& + \left\{ \left\{ \frac{-R_2}{(x_{22}R_1 - x_{12}R_2)} + \frac{-x_{21}R_1 + x_{11}R_2 + x_{31}R_2}{(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2} \right\} \times \right. \\
& \left. \left\{ \frac{x_{11}\hat{X}_{22}}{x_{11}x_{22} - x_{12}x_{21}} \right\} \right\} \hat{X}_{31}\ \mathrm{Var}(x_{12}) \\
& + \left\{ \frac{x_{32}\,(x_{21}R_1 + x_{31}R_1 - x_{11}R_2)\,\hat{X}_{31}}{x_{31}\left[(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2\right]} \times \right. \\
& \left. \frac{x_{12}\hat{X}_{11}}{x_{11}x_{22} - x_{12}x_{21}} \right\} \mathrm{Var}(x_{21}) \\
& + \left\{ \left\{ \frac{R_1}{(x_{22}R_1 - x_{12}R_2)} + \frac{-x_{21}R_1 - x_{31}R_1 + x_{11}R_2}{(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2} \right\} \times \right. \\
& \left. \left\{ \frac{-x_{11}\hat{X}_{12}}{x_{11}x_{22} - x_{12}x_{21}} \right\} \right\} \hat{X}_{31}\ \mathrm{Var}(x_{22}) \\
& + \left\{ \left\{ \frac{x_{32}\,(x_{21}R_1 + x_{31}R_1 - x_{11}R_2)\,\hat{X}_{31}}{x_{31}\left[(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2\right]} \times \frac{-x_{12}\hat{X}_{21}}{x_{11}x_{22} - x_{12}x_{21}} \right\} \right. \\
& \quad + \left\{ \frac{x_{32}\,(x_{21}R_1 - x_{11}R_2 - x_{31}R_2)\,\hat{X}_{31}}{x_{31}\left[(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2\right]} \times \right. \\
& \left. \left. \frac{x_{12}\hat{X}_{11}}{x_{11}x_{22} - x_{12}x_{21}} \right\} \right\} \mathrm{Cov}(x_{11},x_{21})
\end{aligned}
$$

$$+ \left\{\left\{\frac{-R_2}{(x_{22}R_1 - x_{12}R_2)} + \frac{-x_{21}R_1 + x_{11}R_2 + x_{31}R_2}{(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2}\right\} \times\right.$$

$$\left\{\frac{-x_{i1}\hat{X}_{12}}{x_{11}x_{22} - x_{12}x_{21}}\right\} +$$

$$\left\{\frac{R_1}{(x_{22}R_1 - x_{12}R_2)} + \frac{-x_{21}R_1 - x_{31}R_1 + x_{11}R_2}{(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2}\right\} \times$$

$$\left.\left\{\frac{x_{i1}\hat{X}_{22}}{x_{11}x_{22} - x_{12}x_{21}}\right\}\right\} \hat{X}_{31}\ \mathrm{Cov}(x_{12},x_{22}), \quad i=1,2$$

$$\mathrm{Cov}(\hat{X}_{i1},\hat{\lambda}_3) = \left\{\frac{-x_{i2}\hat{X}_{21}\,(-x_{22}R_1 + x_{12}R_2 + x_{32}R_2 - x_{22}R_3\hat{\lambda}_3)}{(x_{11}x_{22} - x_{12}x_{21})^2\, R_3}\right\} \mathrm{Var}(x_{11})$$

$$+ \left\{\frac{x_{i1}\hat{X}_{22}\,(x_{21}R_1 - x_{11}R_2 - x_{31}R_2 + x_{21}R_3\hat{\lambda}_3)}{(x_{11}x_{22} - x_{12}x_{21})^2\, R_3}\right\} \mathrm{Var}(x_{12})$$

$$+ \left\{\frac{x_{i2}\hat{X}_{11}\,(-x_{22}R_1 - x_{32}R_1 + x_{12}R_2 + x_{12}R_3\hat{\lambda}_3)}{(x_{11}x_{22} - x_{12}x_{21})^2\, R_3}\right\} \mathrm{Var}(x_{21})$$

$$+ \left\{\frac{-x_{i1}\hat{X}_{12}\,(x_{21}R_1 + x_{31}R_1 - x_{11}R_2 - x_{11}R_3\hat{\lambda}_3)}{(x_{11}x_{22} - x_{12}x_{21})^2\, R_3}\right\} \mathrm{Var}(x_{22})$$

$$+ \left\{\frac{-x_{i2}\hat{X}_{21}\,(-x_{22}R_1 - x_{32}R_1 + x_{12}R_2 + x_{12}R_3\hat{\lambda}_3)}{(x_{11}x_{22} - x_{12}x_{21})^2\, R_3} + \frac{x_{i2}\hat{X}_{11}\,(-x_{22}R_1 + x_{12}R_2 + x_{32}R_2 - x_{22}R_3\hat{\lambda}_3)}{(x_{11}x_{22} - x_{12}x_{21})^2\, R_3}\right\} \mathrm{Cov}(x_{11},x_{21})$$

$$+ \left\{\frac{x_{i1}\hat{X}_{22}\,(x_{21}R_1 + x_{31}R_1 - x_{11}R_2 - x_{11}R_3\hat{\lambda}_3)}{(x_{11}x_{22} - x_{12}x_{21})^2\, R_3} + \frac{-x_{i1}\hat{X}_{12}\,(x_{21}R_1 - x_{11}R_2 - x_{31}R_2 + x_{21}R_3\hat{\lambda}_3)}{(x_{11}x_{22} - x_{12}x_{21})^2\, R_3}\right\}\Bigg\} \mathrm{Cov}(x_{12},x_{22})$$

$i = 1,2$

$$\mathrm{Cov}(\hat{X}_{31},\hat{\lambda}_3) = \left[\frac{x_{32}\,(x_{21}R_1 - x_{11}R_2 - x_{31}R_2)\,\hat{X}_{31}}{x_{31}\left[(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2\right]} \times\right.$$

$$\left.\frac{-x_{22}R_1 + x_{12}R_2 + x_{32}R_2 - x_{22}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\,R_3}\right]\mathrm{Var}(x_{11})$$

$$+\left[\frac{-R_2}{(x_{22}R_1 - x_{12}R_2)} + \frac{-x_{21}R_1 + x_{11}R_2 + x_{31}R_2}{(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2} \times\right.$$

$$\left.\frac{x_{21}R_1 - x_{11}R_2 - x_{31}R_2 + x_{21}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\,R_3}\right]\hat{X}_{31}\ \mathrm{Var}(x_{12})$$

$$+\left[\frac{x_{32}\,(x_{21}R_1 + x_{31}R_1 - x_{11}R_2)\,\hat{X}_{31}}{x_{31}\left[(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2\right]} \times\right.$$

$$\left.\frac{-x_{22}R_1 - x_{32}R_1 + x_{12}R_2 + x_{12}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\,R_3}\right]\mathrm{Var}(x_{21})$$

$$+\left[\frac{R_1}{(x_{22}R_1 - x_{12}R_2)} + \frac{-x_{21}R_1 - x_{31}R_1 + x_{11}R_2}{(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2} \times\right.$$

$$\left.\frac{x_{21}R_1 + x_{31}R_1 - x_{11}R_2 - x_{11}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\,R_3}\right]\hat{X}_{31}\ \mathrm{Var}(x_{22})$$

$$+\left[\left\{\frac{x_{32}\,(x_{21}R_1 - x_{11}R_2 - x_{31}R_2)\,\hat{X}_{31}}{x_{31}\left[(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2\right]} \times\right.\right.$$

$$\left.\frac{-x_{22}R_1 - x_{32}R_1 + x_{12}R_2 + x_{12}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\,R_3}\right\}$$

$$+\left\{\frac{x_{32}\,(x_{21}R_1 + x_{31}R_1 - x_{11}R_2)\,\hat{X}_{31}}{x_{31}\left[(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2\right]} \times\right.$$

$$\left.\left.\frac{-x_{22}R_1 + x_{12}R_2 + x_{32}R_2 - x_{22}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\,R_3}\right\}\right]\mathrm{Cov}(x_{11},x_{21})$$

$$+\left[\left\{\frac{-R_2}{(x_{22}R_1 - x_{12}R_2)} + \frac{-x_{21}R_1 + x_{11}R_2 + x_{31}R_2}{(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2}\right\} \times\right.$$

$$\left\{\frac{x_{21}R_1 + x_{31}R_1 - x_{11}R_2 - x_{11}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\,R_3}\right\}$$

$$+\left\{\frac{R_1}{(x_{22}R_1 - x_{12}R_2)} + \frac{-x_{21}R_1 - x_{31}R_1 + x_{11}R_2}{(x_{22}x_{31} - x_{21}x_{32})\,R_1 + (x_{11}x_{32} - x_{12}x_{31})\,R_2}\right\} \times$$

$$\left.\left\{\frac{x_{21}R_1 - x_{11}R_2 - x_{31}R_2 + x_{21}R_3\hat{\lambda}_3}{(x_{11}x_{22} - x_{12}x_{21})\,R_3}\right\}\right]\,\bar{X}_{31}\ \mathrm{Cov}(x_{12},x_{22})$$

Appendix 2

Figure 1 contains the SAS code for iteratively reweighted nonlinear least squares estimation in the example of Section 5. The data for the first and second samples are entered in lines 50 and 60, respectively. The first 3 entries on each of these lines are the sample counts for each subclass. The last 3 entries are the cumulative removals from each of the subclasses up to the sample time. The DATA step is used to input the data and to create the 2 conditionally independent univariate observations from each sample required for the regression. The starting values for the parameters are set in line 90. As it appears in Figure 1, LAMBDA3 is fixed with a value of 1 and this code will produce estimates under the Otis (1980) model. Estimates under the new model can be obtained with this code by removing the semicolon separating LAMBDA3 from the parameter list. This will result in LAMBDA3 being considered as a parameter with a starting value of 1.

```
DATA A;
 INPUT X1JSAMP X2JSAMP X3JSAMP R1CUM R2CUM R3CUM;
 I='X1OBS'; OUTPUT; I='X2OBS'; OUTPUT;
 CARDS;
 128 119 253 0 0 0
 227 167 106 140 280 560
 ;
PROC NLIN METHUD=DUD SIGSQ=1;
 PARAMETERS X11=700 X21=700 X31=700; LAMBDA3=1;
 X=X11-R1CUM; Y=X21-R2CUM;
 N=X11-R1CUM+X21-R2CUM+LAMBDA3*(X31-R3CUM);
 IF I='X2OBS' THEN DO;
  CNT=X2JSAMP; NUM=X1JSAMP+X2JSAMP+X3JSAMP; PRB=Y/N; END;
 ELSE DO;
  CNT=X1JSAMP; NUM=X1JSAMP+X3JSAMP; PRB=X/(N-Y); END;
MODEL CNT=NUM*PRB;
_WEIGHT_=1/(NUM*PRB*(1-PRB));
```

Figure 1. SAS code for obtaining estimates in the example of Section 5.