#1937

A NOTE ON NEURONAL FIRING AND INPUT VARIABILITY

Charles E. Smith

Biomathematics Program

Department of Statistics

North Carolina State University

Raleigh, North Carolina 27695-8203

Running Title: Neuronal Firing and Input Variability

1. INTRODUCTION

The subthreshold depolarization of a nerve cell receiving a multitude of random synaptic inputs has often been modeled as an Ornstein-Uhlenbeck (O-U) process (see Tuckwell, 1988 for references). Wan and Tuckwell (1982) obtained asymptotic results for the mean and variance of the interspike intervals for the O-U model with a constant threshold for firing, θ. They used perturbation methods and the method of matched asymptotic expansions on the boundary value problem (Darling and Siegert, 1953) for the moments of the interspike intervals. For the case of small synaptic noise and deterministic crossings, their case (A), we compare their results for the mean and variance to those obtained from Stein's approximation method (Stein, 1967, p 53), a slight generalization of Stein's method and a Wiener process with linear drift approximation. The resultant expression for the variance in Stein's method is identical to that of Wan and Tuckwell. Stein's approximation method is based on long correlation times of the voltage process relative to the standard deviation of the interspike intervals, while the Wiener process with drift is at the other extreme in that its covariance is delta correlated.

Following the notation of Wan and Tuckwell, the subthreshold voltage, $X(t)$, satisfies the linear stochastic differential equation

$$dX(t) = [-X(t) + a]\, dt + b\, dW(t), \tag{1.1}$$

where time is measured in units of the membrane time constant, $W(t)$ is a standard Wiener process, a is a constant representing the mean synaptic input and b scales the Wiener process. The initial condition is taken to be $X(0) = 0$, corresponding to the neuron resetting to the resting level following an action potential. We are interested in the distribution or moments of the time T at which $X(t)$ first reaches the threshold θ. If voltages are measured in units of the threshold θ, equation (1) becomes

$$dY(t) = [-Y(t) + \alpha]\, dt + \epsilon\, dW(t), \tag{1.2}$$

where $Y(t) = X(t)/\theta$, $\alpha = a/\theta$, and $\epsilon = b/\theta$. The process $Y(t)$ is an Ornstein-Uhlenbeck process with a steady state Gaussian distribution of mean α and standard deviation ϵ. Due to the normalization the threshold now has a value of 1.

Wan and Tuckwell's case (A) requires small noise, $\epsilon^2 \ll 1$, and that the mean steady state response, α, be above threshold, i. e., $\alpha - 1 \gg \epsilon$. The approximate expressions for the mean and variance of the firing time were found to be (their equations (6) and (7))

$$E[T] \sim \log\left(\frac{\alpha}{\alpha - 1}\right) - \frac{\epsilon^2}{4}\left(\frac{1}{(\alpha - 1)^2} - \frac{1}{\alpha^2}\right) + O(\epsilon^4) \tag{1.3}$$

$$\mathrm{var}[T] \sim \frac{\epsilon^2}{2}\left(\frac{1}{(\alpha - 1)^2} - \frac{1}{\alpha^2}\right) + O(\epsilon^4) \tag{1.4}$$

The coefficient of variation, C.V., is found by dividing the standard deviation by the mean. In the limit of large mean input, $\alpha \gg 1$, they found (appendix, equation (51))

$$\mathrm{C.V.} \sim \frac{\epsilon}{\sqrt{\alpha}}\left(1 - \frac{1}{4\alpha}\right). \tag{1.5}$$

2. STEIN'S APPROXIMATION METHOD AND A GENERALIZATION

Using a geometric argument, Stein (1967, p53) presented an approximation method for the mean and variance of the interspike intervals in a Poisson driven version of equation (1.1). He was concerned with the scenario in which fluctuations in the first passage time (FPT) to a threshold S(t) are small perturbations about the time, t^*, at which the mean voltage $\mu(t)$ crosses S(t). Our description is a slight generalization of his method. Let $r(t) = S(t) - \mu(t)$, which can be thought of as a recovery process following a spike. Besides requiring that the mean voltage crosses S(t), i.e. deterministic crossings, we require that: (1) the voltage distribution doesn't change its shape drastically near t^*, (2) the resultant standard deviation of the interspike intervals, σ_t, is considerably less than the dominant time constant of the voltage process's autocorrelation around t^*, (3) r(t) is invertible and sufficiently smooth around t^*. Let h be the inverse function of r(t), that is $h(r(t)) = t$. Then the probability density function (p.d.f.) of the crossing times is approximately $f(x)/|dh(x)/dx|$ evaluated at t^*, where f(x) is the marginal p.d.f. of the voltage process X(t). This is the usual Jacobian transformation of a random variable. The conditions above are requiring that around t^* the voltage process, with its timevarying mean subtracted out, behaves like a random variable, i.e. a singular random process. These conditions are more restrictive than Stein's in that he only required the recovery process to be nearly linear within a few standard deviations of t^* in condition (3).

Often only the first few moments of the FPT are of interest. They can be approximated by using a Taylor series expansion of h about $r(t^*)$ (Papoulis, 1965, pp 151-152). The approximations for the mean and variance are given by

$$E[T] \approx t^* + h'' \mu_2/2 + h''' \mu_3/6 + \ldots \qquad (2.1)$$

$$Var[T] \approx \left[h'\right]^2 \mu_2 + h'h'' \mu_3 - \left[h''\mu_2/2 + h'''\mu_3/6\right]^2 + \ldots \qquad (2.2)$$

where prime denotes differentiation with respect to voltage and evaluated at $r(t^*)$, and μ_n is the nth central moment of $X(t^*)$. Recall that the first derivative of an inverse function can be expressed as the reciprocal of the derivative of the original function, so that

$$h' = \left[dr(t^*)/dt\right]^{-1} = \left[dS(t)/dt - d\mu(t)/dt\right]^{-1}\Big|_{t=t^*} \qquad (2.3)$$

which is the reciprocal of the derivative of the recovery process. The higher order implicit derviatives are harder to interpret in a geometric manner, for example

$$h'' = -\left(\frac{d^2r}{dt^2}\right)\left(\frac{dr}{dt}\right)^{-3}\Bigg|_{t=t^*} . \qquad (2.4)$$

Stein's approximation method gives the first term on the right hand side of (2.1) and (2.2). Note that only h' is required, while the generalization which treats the unrestricted voltage process as a singular random process requires higher order derviatives.

Turning our attention back to the normalized process $Y(t)$ of (1.2), we find t^* from $E(Y(t)) = 1$, to obtain

$$t^* = \log\left(\frac{\alpha}{\alpha - 1}\right) . \qquad (2.4)$$

To obtain expressions to order ϵ^2 for the mean and variance, we evaluate the following

$$\mu(t) = \alpha\ (1 - e^{-t})$$

$$\mu_2(t^*) = \frac{\epsilon^2}{2}(1 - e^{-2t})\Big|_{t=t^*} = \frac{\epsilon^2}{2}\left[1 - \left(\frac{\alpha - 1}{\alpha}\right)^2\right]$$

$$h'(r(t^*)) = -(\alpha-1)^{-1} \qquad (2.5)$$

$$h''(r(t^*)) = (\alpha-1)^{-2}$$

Substituting these expressions into (2.1) and (2.2), we find that the first two terms of (2.1) now match Wan and Tuckwell's result (1.3) to order ϵ^2 except for the wrong sign on the second term. The first term of (2.2), Stein's approximation, matches (1.4) to order ϵ^2. Stein's mean approximation is simply t^* and agrees with the first term of (1.3). The generalization by treating the problem as a transformation of a random variable produces a qualitatively different effect on the mean interval due to noise. In Wan and Tuckwell's expression (1.3) the mean is shortened, while the generalization erroneously produces a lengthening effect of the same magnitude. The C.V. produced by Stein's method and its generalization is thus smaller than that in (1.5). The next section provides a bound for the C.V. that is an overestimate.

3. APPROXIMATION BY WIENER PROCESS WITH DRIFT

The previous section was concerned with the limiting case of infinite correlation times, another limiting case is obtained by deleting the $-Y(t)$ term on the right hand side of (1.2). The process is now a Wiener process scaled by ϵ and with a positive drift of αt. The first passage time distribution to the normalized threshold at 1 will now be an inverse Gaussian distribution (see for example, Lerche, 1986). The mean, variance and C.V. are

$$E[T] = 1/\alpha,$$

$$Var[T] = \epsilon^2 \alpha^{-3},$$

$$C.V. = \epsilon/\sqrt{\alpha} .$$

Note that this limiting case provides a shorter mean interval and larger C.V. than for the O-U expressions of section 1. The condition for (1.5) that $\alpha \gg 1$ means that the threshold value is quite low and the mean and variance of the voltage process are close to linear below threshold. While these moments of the O-U process are behaving like those of our Wiener process with drift, the correlation time for O-U process is much longer. In this sense the Wiener process with drift provides another limiting case that has easy-to-calculate results for the moments.

In summary, we have shown that in the small noise, large threshold case some simple approximation methods for the moments of the O-U first passage time provide bounds for those obtained via a more detailed perturbation analysis. For an appropriate range of parameters, Stein's method can also provide good approximations for the mean and variance of more complicated neural models with discontinuous trajectories, e. g. (Smith and Smith, 1984), where a complete perturbation analysis is even more difficult.

Acknowledgements

Support for this work was provided in part by Office of Naval Research Contract N00014-85-K-0105.

REFERENCES

Darling, D. A .D. and Siegert, A. J. F. (1953). The first passage time problem for a continuous Markov process. Ann. Math. Stat. 24, 624-639.

Lerche, H. R. (1986). Boundary Crossing of Brownian Motion, Springer-Verlag, Berlin.

Papoulis, A. (1965). Probability, Random Variables, and Stochastic Processes, 151-152, McGraw-Hill, New York.

Smith, C. E. and Smith, M. V. (1984). Moments of voltage trajectories for Stein's model with synaptic reversal potentials. J. Theoret. Neurobiol. 3, 67-77.

Stein, R. B. (1967). Some models of neuronal variability. Biophys. J. 7, 38-68.

Tuckwell, H. C. (1988). Introduction to Theoretical Neurobiology, Vol. 2.: Nonlinear and Stochastic Theories., Cambridge University Press, Cambridge.

Wan, F. Y. M. and Tuckwell, H. C. (1982). Neuronal firing and input variability. J. Theoret. Neurobiol. 1, 197-218.

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public Release: Distribution Unlimited
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): MIMEO SERIES NO. 1937	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION: North Carolina State Univ. — 6b. OFFICE SYMBOL (If applicable): 4B855	7a. NAME OF MONITORING ORGANIZATION: Office of Naval Research, Department of the Navy
6c. ADDRESS (City, State, and ZIP Code): Dept. of Statistics, Raleigh, North Carolina 27695-8203	7b. ADDRESS (City, State, and ZIP Code): 800 North Quincy Street, Arlington, Virginia 22217-5000
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: Office of Naval Research — 8b. OFFICE SYMBOL (If applicable): ONR	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: N00014-85-K-0105
8c. ADDRESS (City, State, and ZIP Code): Department of the Navy, 800 North Quincy Street, Arlington, Virginia 22217-5000	10. SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO. / PROJECT NO. / TASK NO. / WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification): A NOTE ON NEURONAL FIRING AND INPUT VARIABILITY. (UNCLASSIFIED).

12. PERSONAL AUTHOR(S): Charles E. Smith

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
TECHNICAL	FROM ______ TO ______	January 1989	9

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

ABSTRACT

The Ornstein-Uhlenbeck process with a constant forcing function has often been used as a model for the subthreshold membrane potential of a neuron. The mean, variance and coefficient of variation of the first passage time to a constant threshold are examined for this model in the limit of small synaptic noise and low thresholds. A comparison is made between the asymptotic results of Wan and Tuckwell, who used perturbation analysis and the method of matched asymptotic expansion, and several computational simplier approximation methods. A slight generalization of Stein's method leads to an overestimate of the mean interval while an approximation by a Wiener process with linear drift leads to an underestimate of the true mean interval. These bounds are simple to calculate and can be used as a prelude to the more detailed analysis of perturbation theory.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: ☒ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE (Include Area Code) / 22c. OFFICE SYMBOL