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A RELIABILITY APPLICATION OF

THE MIXTURE OF INVERSE GAUSSIAN DISTRIBUTIONS

Charles E. Smith[1] and Petr Lánský[2]

Biomathematics Program
Dept. of Statistics
Campus Box 8203
North Carolina State Univ.
Raleigh, North Carolina 27695-8203, USA

1 Author to whom correspondence should be addressed. Supported in part by Office of Naval Research Contract N00014-85-K-0105 and NAS-CSAV exchange scientist program.

2 Permanent address: Inst. Physiol., Czech. Acad. Sci., Prague, Czechoslovakia.

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SUMMARY & CONCLUSIONS: A mixture of Inverse Gaussian distributions is examined as a model for the lifetime of components that differ in their initial quality. The interpretation of the Inverse Gaussian distribution as the first passage time of Brownian motion with positive drift is adopted. Two cases are studied: initial quality of two levels and a continuous uniform distribution of initial quality. Parameter estimation is examined for these two cases. The model is most appropriate for those cases where the Inverse Gaussian fails due to heterogeneity of the initial component quality.

1. Introduction.

The Inverse Gaussian distribution (IGD) has been proposed and examined several times as a lifetime model (e.g., [2], [5], [9]). It is particularly useful when the lifetime distribution reflects an initial high rate of wear and failure via an early mode and positive skew; and the hazard rate first increases and then decreases to a nonzero asymptotic level. One of its advantages over other lifetime models follows from its mechanistic interpretation as the first-passage-time of Brownian motion across a constant boundary, S ([4], [6], [10]). In this interpretation the introduction of a random initial condition, X_0, can be viewed as a different quality assigned to each item at the moment of its production, and that subsequent changes in quality (cumulative wear, fatigue, crack growth, etc.) can be modeled as a Wiener process with positive drift (see e.g. [4]). Denoting this process by X(t) and the initial value X(0) by X_0, then $P(X(0) = x_0 > S)$ becomes the probability that a new item is a defective one at the moment of its production. For the sake of simplicity we further assume that $P(X(0) > S) = 0$ or that this probability is negligible. In order to retain the physical interpretation of the model mentioned above, we will use the parameterization from the diffusion-threshold viewpoint rather than that commonly used for the IGD (c.f., [3]).

Notation

$\sim$	implies: is distributed as
α,β	mean and shape parameters of Inverse Gaussian distribution
$IGD(\alpha,\beta)$	implies an Inverse Gaussian r.v. with parameters α and β
$f(t;S,x_0,\mu,\sigma^2)$	pdf of the first passage time of Wiener process with drift μ
$h(t;\alpha,\beta)$	pdf of Inverse Gaussian distribution
X_0	r.v. characterizing the initial condition
$w(X_0)$	pdf of r.v. X_0
CV	implies coefficient of variation, $CV = \frac{\sqrt{Var}}{E}$
$\hat{}$	implies a maximum likelihood estimator

For other standard notation see "Information for Readers & Authors" at the back of each issue.

2. Fixed initial condition.

The first passage time of a Wiener process with drift $\mu > 0$ and infinitesimal variance $\sigma^2 > 0$ through a constant boundary S, under the condition that the process starts at $x_0 < S$ at time zero, is a r.v. T with pdf

$$f(t;S,x_o,\mu,\sigma^2) = \frac{|S-x_0|}{\sqrt{2\pi\sigma^2 t^3}} \exp\left(- \frac{(S - x_o - \mu t)^2}{2\sigma^2 t}\right) \tag{1}$$

Using the transformation $\alpha = (S - x_0)/\mu$ and $\beta = (S - x_0)^2/\sigma^2$ then $T \sim IG(\alpha,\beta)$ with pdf

$$h(t;\alpha,\beta) = \left(\frac{\beta}{2\pi t^3}\right)^{\frac{1}{2}} \exp\left(- \frac{\beta(t - \alpha)^2}{2\alpha^2 t}\right), \tag{2}$$

and using this transformation the results presented here can be compared with those given in the literature cited above. A lower value of x_0 in the model (1) can be interpreted as better initial quality and thus longer expected lifetime. Note that a change in x_0 causes a change in both of the parameters α and β. For instance, as x_0 approaches the threshold S, with μ and σ fixed, the pdf becomes more positively skewed and the mode and mean approach a value of zero. On the other hand, as the initial quality becomes increasingly better, the pdf becomes more s-normal in shape but with an increasing mean and variance.

3. Two levels of initial quality.

Let us assume that x_0 is replaced by the discrete r.v. X_0 for which $P(X_0 = x_{0,1}) = p$ and $P(X_0 = x_{0,2}) = 1 - p$. Then the density of the lifetime distribution g(t) is a mixture of densities; expressed in terms of (1) it is

$$g(t) = p\, f(t;S,x_{0,1},\mu,\sigma^2) + (1-p)\, f(t;S,x_{0,2},\mu,\sigma^2) \ . \tag{3}$$

From this fact all the properties of r.v. D distributed in accordance with (3) can be derived

$$E(D) = \frac{1}{\mu}\Big(S - (p\,x_{0,1} + (1\text{-}p)\,x_{0,2})\Big) \tag{4}$$

$$Var(D) = \frac{\sigma^2}{\mu^3}\Big(S - (p\,x_{0,1} + (1\text{-}p)\,x_{0,2})\Big) + \frac{p(1\text{-}p)}{\mu^2}(x_{0,1} - x_{0,2})^2 \tag{5}$$

$$= \mu^{-2}\Big(\sigma^2\, E(D) + p(1\text{-}p)(x_{0,1} - x_{0,2})^2\Big)$$

$$CV^2(D) = \frac{\sigma^2}{\mu}\Big(S - (p\,x_{0,1} + (1\text{-}p)\,x_{0,2})\Big)^{-1} + \frac{p(1\text{-}p)(x_{0,1} - x_{0,2})^2}{\Big(S - (p\,x_{0,1} + (1\text{-}p)\,x_{0,2})\Big)^2}\,. \tag{6}$$

Note that if $\sigma > \mu > S - E(X_0)$, then $CV > 1$ analogous to the fixed initial condition case.

Using (4), (5) and (6), the effect of the variability of the initial condition can be seen by the following comparison of models (1) and (3) with identical parameters μ and σ. Set the value of the fixed initial quality in model (1) equal to the mean initial quality in model (3). Then the mean lifetimes of the two models are identical, but the variance of model (3) is larger than that of model (1) by an amount equal to the second term in (5). The resultant CV in (6) is thus also larger than in model (1) as is intuitively expected. Using (3), we can also compute several other characteristics commonly used in reliability studies. For example, the survival function is

$$Sf(t) = p\left\{gauf\left(\frac{(S - x_{0,1})^2 - t\sigma^2}{\sqrt{\mu(S-x_{0,1})^3 t}}\right) - \exp\left(\frac{2\,\sigma^2}{\mu(S-x_{0,1})}\right) gauf\left(-\frac{(S-x_{0,1})^2 + t\sigma^2}{\sqrt{\mu(S-x_{0,1})^3 t}}\right)\right\} +$$

$$+ (1\text{-}p)\left\{gauf\left(\frac{(S-x_{0,2})^2 - t\sigma^2}{\sqrt{\mu(S-x_{0,2})^3 t}}\right) - \exp\left(\frac{2\sigma^2}{\mu(S-x_{0,2})}\right) gauf\left(-\frac{(S-x_{0,2})^2 + t\sigma^2}{\sqrt{\mu(S-x_{0,2})^3 t}}\right)\right\}, \tag{7}$$

and the hazard rate, hr(t), is obtained by combining (3) and (7), and is notationally complicated but computationally simple. For the model (1) the asymptotic value of hazard rate is $\sigma^4/(2\mu(S\text{-}x_0)^3)$ as t tends to infinity. While for model (3) the asymptotic value of the hazard rate is $\min_i\,(\sigma^4/(2\mu(S\text{-}x_{0,i})^3)$, which is the asymptotic hazard rate of the item with better initial quality, i.e., lower value of X_0.

The properties mentioned above are illustrated graphically in figures (1, 2, 3). For all three plots, the values of S, μ, and σ are fixed at 10, 10, and 20 respectively. The effect of the mixing parameter p on the shape of the pdf (3) is shown in figure 1, a semilog plot of the pdf vs. lifetime with $x_{0,1} = -6$ and $x_{0,2} = +6$. The value 0 is taken here as a reference level for initial quality with positive values, i.e. those closer to the threshold, being of a worse quality than those with negative values. The middle three curves correspond to mixture distributions with p = 0.2, 0.4, and 0.8. The upper curve at early times represents an fixed initial value of + 6, i.e., p = 0 while p = 1 corresponds to the lower curve at early lifetimes and represents a fixed initial value of -6. For the three mixtures the early behaviour of the pdf is dominated by the mode corresponding to the initial condition closest to the threshold, i. e. $x_{0,2}$. All five curves cross at the same point as expected from (3). Only the p = 0.8 curve is bimodal, showing that the position of the lower initial condition can be more difficult to ascertain from a visual inspection of the pdf. The general conditions for bimodality of the pdf are not known. The hazard rate may also be bimodal, but the five hazard rates corresponding to figure 1 will not all intersect at the same value of time.

The decomposition of the mixture's pdf and hazard rate into two components is illustrated in figure 2 for two sets of initial conditions. In A the mixture pdf with p = 0.8 of figure 1 is shown along with the two pdf's corresponding to a fixed initial value at $x_{0,1} = -6$ and $x_{0,2} = 6$. While 2A is clearly bimodal, reducing the spread of the two initial values produces a mixture in figure 2B that is barely bimodal even though the position of the early mode has only shifted slightly from A. In figure 2C,D, the corresponding hazard rates are shown. Unlike the pdf's, the hazard rates do not decompose precisely into the sum of two components corresponding to fixed initial values at $x_{0,1}$ and $x_{0,2}$.

The final set of figures (figure 3) compare the mixture's pdf and hazard rate to the corresponding curves from a single initial value. The single initial value is equal to the mean of the mixture's initial value. Figures 3A and 3B show that the most dramatic differences occur at early times for both the pdf and the hazard rate. Increasing the spread of the initial values from ± 4 to ± 6 produces even more pronounced differences at early lifetimes or failure times (figure 3 C,D). Here the effect of model misspecification has some practical reliability consequences. The plots for the mixture in 3C,D might suggest a burn-in procedure for the product, while the corresponding curves for a single initial value would not. As noted above, the variance for the mixture distribution is larger than that for the corresponding fixed-initial-value curve. However, it is difficult to visually notice this increase in figure 3 (2% in A, 4% in C).

We now turn to the problem of parameter estimation. Amoh [1] developed iterative procedures for maximum likelihood estimation of parameters in a mixture of two IGD's. He estimated the means and mixing proportion under the condition that the two Inverse Gaussian populations had a common and known shape parameter (β of (2)). This condition is not met in our problem and an extension of his iterative maximum likelihood procedure appears to be quite complicated numerically. On the other hand, if the initial quality can be measured directly, i. e., completely classified samples in Amoh's terminology, then the mixing proportion p is easily estimated as the relative frequency of $x_{0,1}$. Using this estimated proportion, procedures for estimating μ and σ are well known [3]. If the values of X_0 are not measured directly, then the method of moments could be used to estimate p, μ, and σ under the condition that S is known ([8], p.200).

At a more qualitative level, as noted by Amoh [1], it is difficult to decompose the mixture into its components if $x_{0,1}/\mu \approx x_{0,2}/\mu$. Difficult in the sense that a much larger sample size is required to distinguish this case from a fixed initial condition. On the other hand, such a misspecification of the model does not produce large differences between the shapes of the corresponding pdf's.

4. Continuously changing initial quality.

Let us assume now that the initial condition X_0 is a r.v. with pdf $w(x_0)$ defined on (-

∞,S) . Then the pdf g(t) can be computed from the relationship

$$g(t) = \int_{-\infty}^{S} f(t;S,x_0,\mu,\sigma^2)w(x_0)dx_0 \ . \tag{8}$$

A uniform distribution of X_0 over ($x_{0,min}$, S) can be interpreted as one type of controlled production within a set of tolerance limits, where $x_{0,min}$ is the minimum initial wear, i. e. best initial quality. Substituting

$$w(x_0) = (S-x_{0,min})^{-1} \tag{9}$$

into (8) we obtain

$$g(t) = \left(t(S-x_{0,min})\right)^{-1}E(Y), \tag{10}$$

where Y is s-normal with mean μt and variance $\sigma^2 t$ truncated at 0 and ($S-x_{0,min}$) and not normalized to be a proper pdf on this interval.

A similar expression to (10) is obtained if we take the initial distribution to be uniform over the range ($x_{0,min}$, $x_{0,max}$) with $x_{0,max} < S$. The interpretation is that we now have better control over the initial quality. For other types of distributions for X_0 (s-normal, resp. truncated s-normal) (8) must be calculated numerically.

Some method of parameter estimation is required for model validation. Maximum likelihood estimates of the parameters in the density (8) can be easily computed when the observed data consists of the pairs (x,t) = (x_i,t_i, i=1,...,N) i.e., measurements of the initial quality, x_i, and the lifetime, t_i, on N independent samples. They are

$$\hat{\mu} = \sum_{i=1}^{N} (S-x_i) \,/ \sum_{i=1}^{N} t_i \quad , \tag{11}$$

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{N} \frac{(S-x_i)^2}{t_i} - \sum_{i=1}^{N} (S-x_i)^2 \,/ \sum_{i=1}^{N} t_i \tag{12}$$

under the assumption that S is known. Note when the initial quality is fixed, i.e., $x_i = x_0$, we get the usual estimates, $\hat{\mu} = (S - x_0)/\overline{t}$ and $\hat{\sigma}^2 = (S - x_0)^2 \left(\overline{(1/t)} - 1/(\overline{t})\right)$ where bar denotes sample mean [3]. It is well known that for fixed x_0 the estimate of μ is biased and the same holds for (11),

$$E(\hat{\mu}) = \mu + E_{X_0}\left(\frac{1}{S - X_0}\right) \ . \tag{14}$$

5. Discussion.

Another way to view the model considered here is that there is heterogeneity among units, with the heterogeneity being solely due to the initial condition. Follmann and Goldberg [7] have recently examined the problem of distinguishing heterogeneity from decreasing hazard rates. They assumed that the failure times for each repairable unit had a Weibull distribution, and that the scale parameter of the Weibull was Gamma distributed across units. The situation for the Inverse Gaussian model is more complicated as the hazard rate with a fixed initial condition is nonmonotonic, the hazard rate first increases and then decreases toward a nonzero asymptotic value. The introduction of heterogeneity may produce bimodal hazard rates and thus further complicate recommendations for inspection times following replacement or burn-in times.

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FIGURE LEGENDS

Figure 1.

Semilog plot of pdf from eqn. (3) for various values of mixture parameter. p is the proportion of $x_{0,1}$ Values of other parameter are $\mu = 10$, $\sigma = 20$, S = 10, $x_{0,1} = -6$ and $x_{0,2} = +6$.

Figure 2.

Decomposition of the mixture's pdf (A,B) and hazard rate (C,D) into two components. $x_{0,1} = -6$ in (A,C) and -4 in (B,D) and with $x_{0,2} = 6$ in (A,C) and 4 in (B,D). Here p = 0.8 and other parameters are as in Figure 1. pdf from equation (3) and hazard rate from ratio of (3) to (7). Solid curves - mixture, dashed curves - components.

Figure 3.

For the same two sets of initial quality as in Figure 2, a comparison of the mixture's pdf and hazard rate is made with that of a single Inverse Gaussian having a fixed initial condition with a value equal to the mean of the mixture's initial quality. $E[X_0] = -2.4$ in (A,B) and -3.6 in (C,D); also p = 0.8, other parameters as in Figure 1. Solid curves correspond to mixture and dashed curves correspond to Inverse Gaussian distribution.

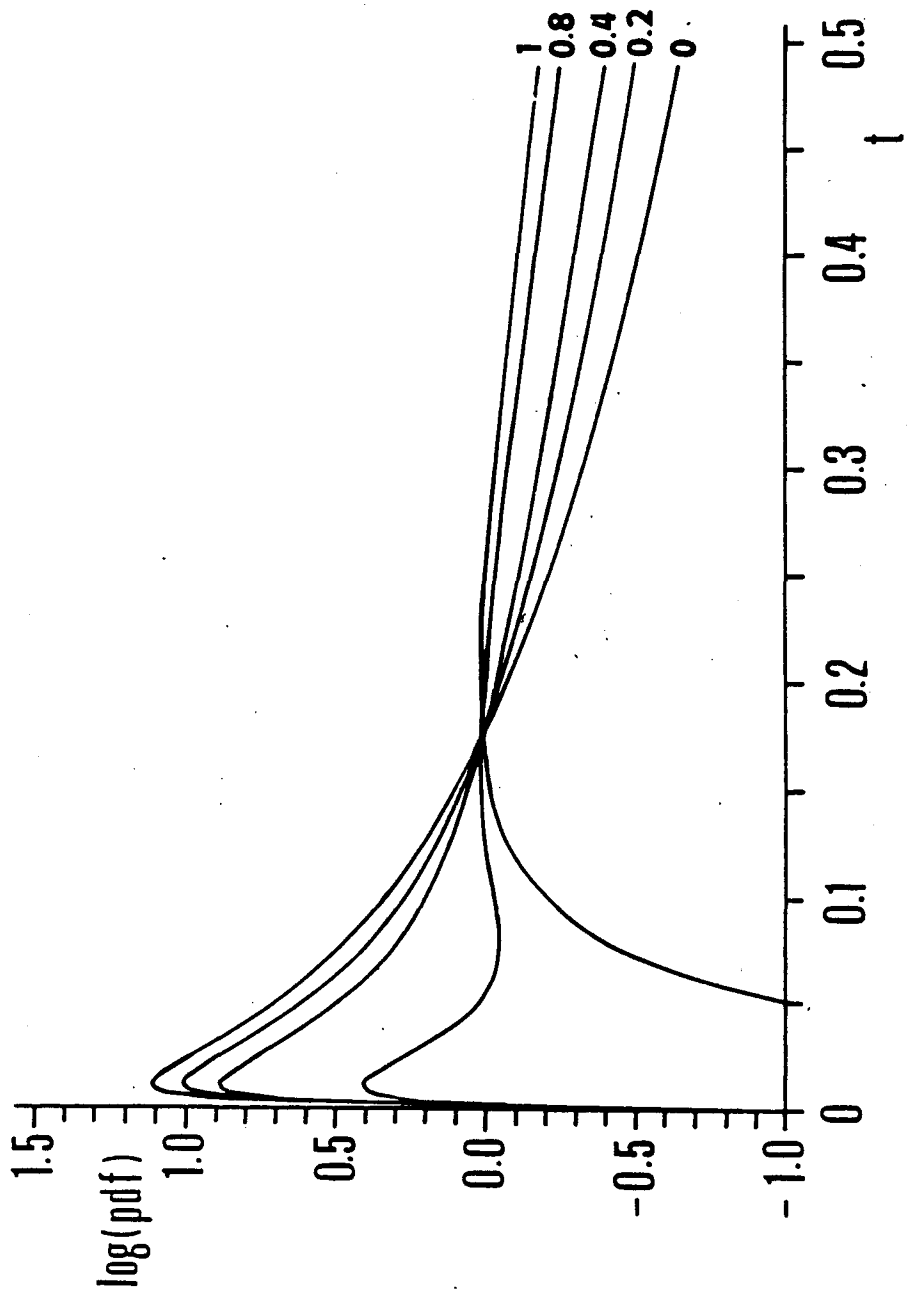

FIGURE 1

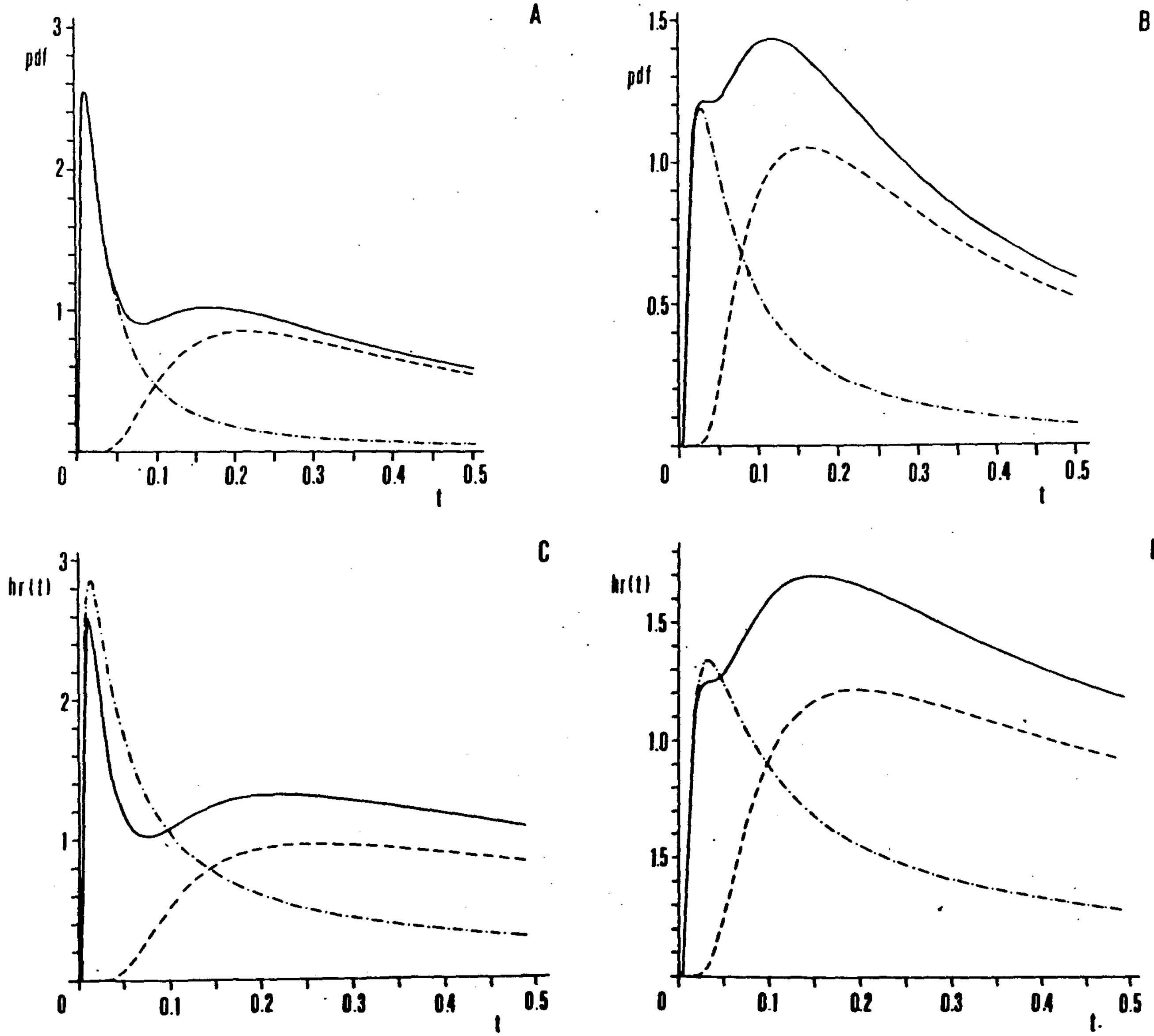

FIGURE 2

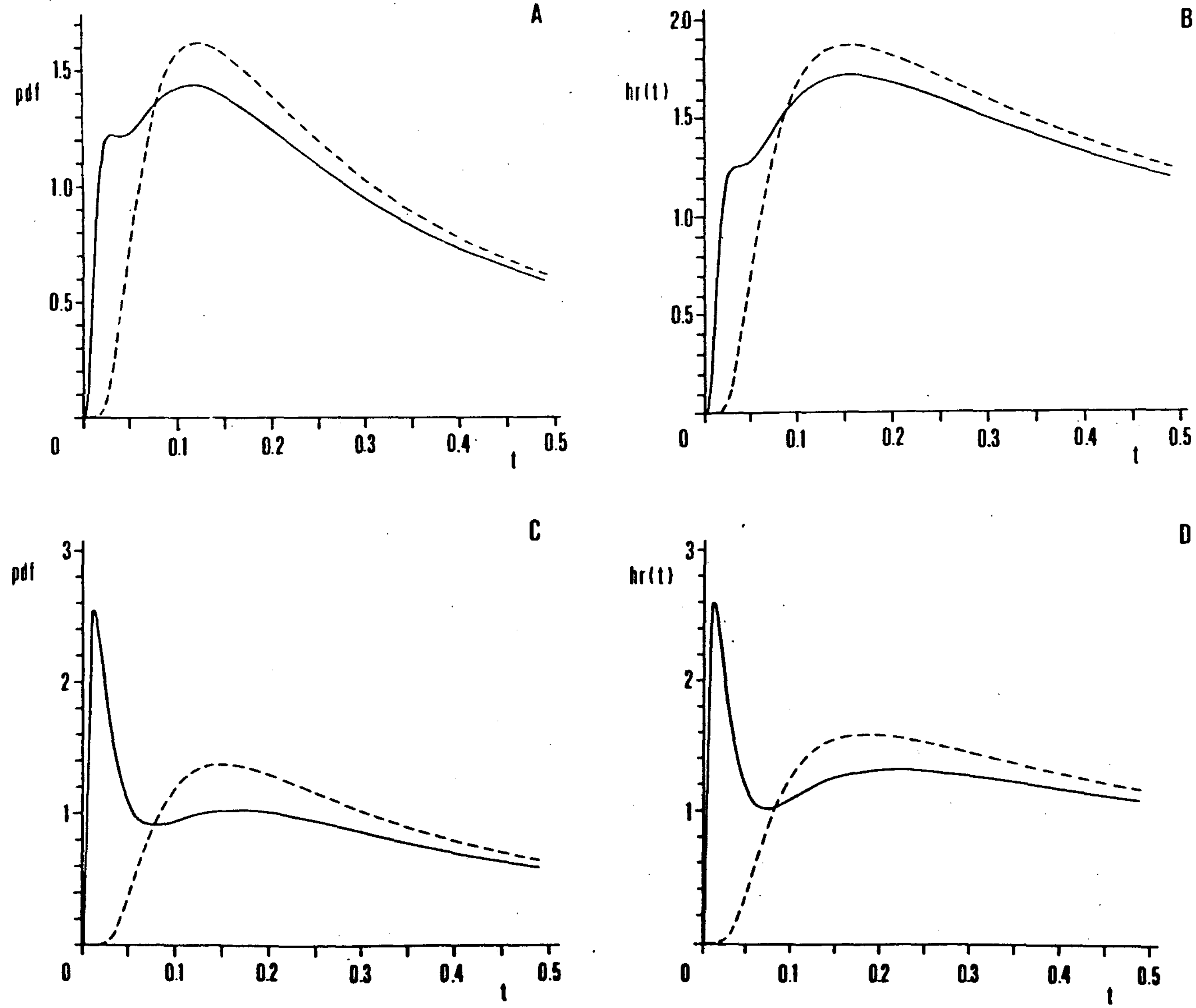

FIGURE 3

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