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INFINITELY DIVISIBLE LAWS PARTIALLY ATTRACTED TO A POISSON LAW

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Groshev, in 1941, derived a necessary and sufficient condition for a fixed infinitely divisible law to belong to the domain of partial attraction of *some* Poisson law. The condition is expressed through the spectral function of the Lévy-Khinchin canonical representation for the characteristic function of the infinitely divisible law in question. The mean λ of the attracting law does not play any role in Groshev's condition. In the present paper we investigate the problem for any fixed $\lambda > 0$ and derive characterizations involving λ in terms of the measure functions of a unique probabilistic representation for an infinitely divisible law, obtained within the framework of a new "probabilistic approach" to sums of independent and identically distributed random variables. Here, as in [2], where an arbitrary law attracted to a Poisson is characterized, the intricate behaviour of the normalizing constants shows up. Depending on whether $\lambda > 1$ or $\lambda < 1$ there are two qualitatively different ways of choosing them and hence the attracting Poisson law arises in two different ways. A concrete construction illustrates these results and shows also that if $\lambda = 1$ then both possibilities may occur*.

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1. INTRODUCTION, THE RESULTS, AND DISCUSSION

Let $F(x) = P\{X \leq x\}$, $x \in R$, be the distribution function of a given random variable X, and let $X_1, X_2, \ldots$ be independent copies of X. We say, that F is partially attracted by the Poisson distribution with mean λ, if there exist a subsequence $\{n'\}$ of the sequence $\{n\}$ of the positive integers and normalizing and centering constants $A_{n'} > 0$ and $C_{n'} \in R$ such that

$$(1.1) \qquad \frac{1}{A_{n'}} \{ \sum_{j=1}^{n'} X_j - C_{n'} \} \to_D Y_\lambda \text{ as } n' \to \infty$$

where $\to_D$ denotes convergence in distribution and Y_λ is a Poisson random variable with mean $\lambda > 0$, that is

$$P\{Y_\lambda = k\} = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0,1,2,\ldots .$$

The distribution functions satisfying (1.1) form the domain of partial attraction of the limiting Poisson law. We write for this set D_p (Poisson (λ)). The domain of partial attraction of any infinitely divisible law G is defined in a way analogous to (1.1). Khinchin proved in 1937 that for every such G the set $D_p(G)$ is not void (cf. [4]). Then, of course, D_p (Poisson (λ)) is not void. It is also known ([4]) that if (1.1) is true, then $\{n'\}$ can not be the whole $\{n\}$; otherwise the limiting distribution should be stable which is not true for the Poisson distribution. (That is exactly the reason that we talk about partial attraction.)

There are many reasons for the interest in the characterization of the distributions partially attracted to a Poisson law. In 1941, Groshev [5] characterized the distribution functions in the set $\bigcup_{\lambda>0} D_p$ (Poisson (λ)). He proved that

$$F \in \bigcup_{\lambda>0} D_p \text{ (Poisson } (\lambda))$$

if and only if

$$(1.2) \qquad \liminf_{h\to\infty} \frac{\int_{|x-1|>\varepsilon} \frac{x^2}{1+x^2}\, dF(hx)}{\int_{|x-1|<\varepsilon} dF(hx)} = 0 \quad \text{for any } \varepsilon > 0.$$

This condition, being in terms of F, does not show immediately in what sense F is restricted just as it does not show at all the role of the mean λ of the attracting law. In the same paper, Groshev also considered a special case of the problem when the underlying F is infinitely divisible and proved that

$$F \in \bigcup_{\lambda>0} D_p \text{ (Poisson } (\lambda))$$

if and only if

$$(1.3) \qquad \liminf_{h\to\infty} \frac{\int_{|u-1|>\varepsilon} \frac{u^2}{1+u^2}\, dG(hu)}{\int_{|u-1|<\varepsilon} dG(hu)} = 0,$$

where G(.) is the Lévy-Khinchin measure function of the canonical form of the characteristic function of F. As in the general case, the condition above completely disguises the role of the parameter λ. Note that although (1.3) and (1.2) are very much alike, the result in the special case is not a direct implication from the result in the general case. To obtain (1.3) Groshev uses the special form of the general criterion of convergence to an infinitely divisible law obtained for the case of an infinitely divisible underlying distribution function (see [4]).

In [2], operating within the framework of the recently developed "probabilistic approach" to sums of independent and identically distributed random variables ([3], [1]) we characterised an arbitrary distribution function F which belongs to D_p (Poisson (λ)). Starting out from the general convergence theorems of the probabilistic approach we delineate the portion of the sample that contributes the limiting Poisson behaviour of the sums in (1.1) and describe the effect of extreme values. This leads to the analytical condition (1.4) below for $F \in D_p$(Poisson (λ)) expressed by the quantile function $Q(s) = \inf\{x: F(x) \geq s\}$, $0 < s < 1$, and the corresponding "truncated variance" function, which by setting $u \wedge v = \min(u,v)$ is

$$\sigma^2(s) = \int_s^{1-s} \int_s^{1-s} (u \wedge v - uv)\, dQ(u)\, dQ(v), \quad 0 < s < 1/2.$$

The condition of [3] is the following:

$$(1.4) \qquad \liminf_{n \to \infty} \frac{\sigma^2(\frac{\varepsilon}{n}) + \sigma^2(\frac{\lambda+\varepsilon}{n}) + \frac{1}{n} Q^2(\frac{\varepsilon}{n})}{\sigma^2(\frac{\lambda-\varepsilon}{n})} = 1 \text{ for } 0 < \varepsilon \leq \lambda/2.$$

This implies that depending on whether $\lambda > 1$ or $\lambda < 1$ there are two principally different ways of choosing the normalizing constants $A_{n'}$ in (1.1). When $\lambda > 1$, then $\delta_1 < \frac{\sqrt{n'}\,\sigma(1/n')}{A_{n'}} < \delta_2$ for two positive δ_1 and δ_2. When $\lambda < 1$, then $\frac{\sqrt{n'}\,\sigma(1/n')}{A_{n'}} \to 0$ as $n' \to \infty$. The case $\lambda = 1$ is a real borderline and both possibilities may occur as shown by the Construction in [3].

Just as the canonical representation of the characteristic function of an infinitely divisible law in (1.5) below is a central point in the classical limit theory, the probabilistic representation of an infinitely divisible law in (1.9) below plays an essential role within the framework of the probabilistic approach. It is desirable then, in case if in (1.1) the underlying law F is infinitely divisible, to have the necessary and sufficient conditions for $F \in D_p(\text{Poisson } (\lambda))$ in terms of the components of the probabilistic representation of F, what is the aim of the present paper. Note that here as in [5], again, a direct implication from the general result (1.4) is not possible.

To formulate the main results we need to introduce the probabilistic representation of an infinitely divisible law. Let $\varphi(t)$, $t \in R$, be the characteristic function of an arbitrary infinitely divisible random variable Y in its Lévy form,

$$(1.5) \qquad \varphi(t) = \exp\{it\theta - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{0} (e^{itx} - 1 - \frac{itx}{1+x^2})dL(x) + \int_{0}^{\infty} (e^{itx} - 1 - \frac{itx}{1+x^2})dR(x)\}.$$

Here θ and $\sigma \geq 0$ are constants and L(x) and R(x) are two non-decreasing uniquely determined left-continuous and right-continuous functions on $(-\infty, 0)$ and $(0, \infty)$, respectively, with $L(-\infty) = R(+\infty) = 0$ such that

$$\int_{-\varepsilon}^{0} x^2 dL(x) + \int_{0}^{\varepsilon} x^2 dR(x) < \infty \quad \text{for any } \varepsilon > 0.$$

Introduce the functions

$$(1.6) \qquad \Psi_1(u) = \inf\{x < 0: L(x) > u\}, \ 0 < u < \infty,$$

$$(1.7) \qquad \Psi_2(u) = \inf\{x < 0: -R(-x) > u\}, \ 0 < u < \infty,$$

which obviously are non-decreasing, non-positive and right-continuous, and also the constant

$$\gamma(\Psi_1, \Psi_2) = \sum_{j=1}^{2} (-1)^{j+1} \{\int_0^1 \frac{\Psi_j(u)}{1+\Psi_j^2(u)} du - \int_1^{\infty} \frac{\Psi_j^3(u)}{1+\Psi_j^2(u)} du\}.$$

Further, let $N_j(\cdot)$, $j = 1,2$ be two independent standard left-continuous Poisson processes with jump points $S_n^{(j)}$, $n = 1,2,\ldots,$ given by

$$N_j(u) = \sum_{n=1}^{\infty} I(S_n^{(j)} < u), \quad 0 \le u < \infty, \quad j = 1,2,$$

where I(.) is the indicator function and with independent exponential random variables $E_1^{(j)}, E_2^{(j)},\ldots,$ all having mean 1,

$$S_n^{(j)} = E_1^{(j)} + \ldots + E_n^{(j)}, \quad n \ge 1, \ j = 1,2.$$

Introduce also the random variables

$$(1.8) \qquad V^{(j)} = (-1)^{j+1}\{ \int_{S_1^{(j)}}^{\infty} (u - N_j(u))d\Psi_j(u) + \int_1^{S_1^{(j)}} u d\Psi_j(u) + \Psi_j(1)\}$$

$$j = 1,2.$$

Let Z be a standard normal variable independent of $(N_1(.), N_2(.))$. Theorem 3 in [3] states that the random variable $V^{(1)} + \sigma Z + V^{(2)} + \theta - \gamma(\Psi_1, \Psi_2)$ has characteristic function $\varphi(t)$ of (1.5), that is we have the distributional equality

$$(1.9) \qquad Y =_D V^{(1)} + \sigma Z + V^{(2)} + \theta - \gamma(\Psi_1, \Psi_2).$$

From now on we assume that the given distribution function F is infinetely divisible. In fact,without loss of generality we can assume that

$$(1.10) \quad X = V^{(1)} + \sigma Z + V^{(2)} := \overline{V}(\Psi_1, \Psi_2, \sigma)$$

which by (1.9) means of course that

$$X =_D Y - \theta + \gamma(\Psi_1, \Psi_2).$$

The simple reason is the following. The condition (1.1) is satisfied if and only if it is also satisfied for independent copies of X - C, with the same $A_{n'}$ and with centering constants $B_{n'} = C_{n'} - n'C$, for any real C. This remark and (1.10) above allow us to write further $(\Psi_1,\Psi_2,\sigma)\in D_p$(Poisson ($\lambda$)) instead of $F\in D_p$(Poisson (λ)), just as it is adopted in [1].

For a positive integer n and for h > 0 we introduce the functions

$$a_1^2(n;h) = n\{\int_{h/n}^{\infty} \Psi_1^2(u)du + \sigma^2 + \int_{h/n}^{\infty} \Psi_2^2(u)du\},$$

$$a_2^2(n;h) = a_1^2(n;h) + \Psi_1^2(h/n) + \Psi_2^2(h/n)$$

and $a_2^2(rn) = ra_2^2(n;\ 1/r)$, $r = 1,2,\ldots$, which are all well defined due to the property that

$$\int_{\varepsilon}^{\infty} \Psi_j^2(u)du < \infty, \quad \varepsilon > 0; \quad j = 1,2,$$

obtained from the square - integrability of L and R.

Let $\lambda > 0$ be fixed and introduce the notation

$$r_1(\lambda) = \min\{r:\ r \text{ positive integer}, \lambda r > 1\}.$$

THEOREM 1. *We have* $(\Psi_1,\Psi_2,\sigma)\in D_p$(Poisson ($\lambda$)) *if and only if for each* $r \geq r_1(\lambda)$ *there exists a genuine subsequence* $\{n'\}$ *of the positive integers such that*

$$(1.11) \qquad \frac{1}{a_2(rn')}\Psi_1\left(\frac{s}{n'}\right) \to 0, \text{ if } s > 0,$$

$$(1.12) \qquad \frac{1}{a_2(rn')}\Psi_2\left(\frac{s}{n'}\right) \to \begin{cases} -\alpha_r, & \text{if } 0 < s < \lambda, \\ 0, & \text{if } s > \lambda, \end{cases}$$

as $n' \to \infty$, *where* $\alpha_r > 0$ *is some constant, and*

$$(1.13) \qquad \lim_{h\to\infty} \limsup_{n'\to\infty} \frac{a_1(n';h)}{a_2(rn')} = 0.$$

To formulate the next theorem we need the function

$$b^2(x;h) = x\{\int_{h/x}^{\infty} \Psi_1^2(u)du + \sigma^2 + \int_{h/x}^{\infty} \Psi_2^2(u)du\} + h\{\Psi_1^2(\tfrac{h}{x}) + \Psi_2^2(\tfrac{h}{x})\}, \quad x > 0.$$

Obviously,

$$(1.14) \quad b^2(n;h) = a_1^2(n;h) + h\{\Psi_1^2(\tfrac{h}{n}) + \Psi_2^2(\tfrac{h}{n})\};$$

$$b^2(n;1) = a_2^2(n;1) = a_2^2(n).$$

THEOREM 2. *The three conditions* (1.11), (1.12) *and* (1.13) *are satisfied along some subsequence* $\{n'\}$ *of the positive integers, and hence* $(\Psi_1,\Psi_2,\sigma)\in D_p$(Poisson (λ)) *if and only if*

$$(1.15) \quad \liminf_{x\uparrow\infty} \frac{b^2(x;\varepsilon) + b^2(x;\lambda+\varepsilon)+\Psi_1^2(\varepsilon/x)}{b^2(x;\lambda-\varepsilon)} = 1 \textit{ for every } 0 < \varepsilon \le \lambda/2.$$

As a matter of fact, instead of (1.15) we will use the condition

$$(1.16) \quad \liminf_{n\to\infty} \frac{b^2(n;\varepsilon) + b^2(n;\lambda+\varepsilon) + \Psi_1^2(\frac{\varepsilon}{n})}{b^2(n;\lambda-\varepsilon)} = 1 \text{ for every } 0 < \varepsilon \le \lambda/2,$$

which is equivalent, as we show just now, to (1.15). Note first that if $h < s$, then

$$b^2(x;h) - b^2(x;s) = x\{\int_{h/x}^{s/x} \Psi_1^2(u)du + \int_{h/x}^{s/x} \Psi_2^2(u)du\}$$

$$+ h\{\Psi_1^2(\tfrac{h}{x}) + \Psi_2^2(\tfrac{h}{x})\} - s\{\Psi_1^2(\tfrac{s}{x}) + \Psi_2^2(\tfrac{s}{x})\}$$

$$\ge x(\tfrac{s}{x} - \tfrac{h}{x})\{\Psi_1^2(\tfrac{s}{x}) + \Psi_2^2(\tfrac{s}{x})\} + h\{\Psi_1^2(\tfrac{h}{x}) + \Psi_2^2(\tfrac{h}{x})\} - s\{\Psi_1^2(\tfrac{s}{x}) + \Psi_2^2(\tfrac{s}{x})\}$$

$$= h\{\Psi_1^2(\tfrac{h}{x}) - \Psi_1^2(\tfrac{s}{x})\} + h\{\Psi_2^2(\tfrac{h}{x}) - \Psi_2^2(\tfrac{s}{x})\} \ge 0,$$

which means that $b^2(x;h)$ is non-increasing as a function of h. Hence

$$\frac{b^2(x;\varepsilon)}{b^2(x;\lambda-\varepsilon)} \geq 1 \text{ for } x > 0 \text{ and } 0 < \varepsilon \leq \lambda/2,$$

and since $\dfrac{b^2(x;\lambda+\varepsilon)}{b^2(x;\lambda-\varepsilon)} \geq 0$, $\dfrac{\Psi_1^2(\frac{\varepsilon}{x})}{b^2(x;\lambda-\varepsilon)} \geq 0$,

(1.15) means that for every $0 < \varepsilon \leq \lambda/2$ there exists a subsequence of real positive numbers $\{x_m\} = \{x_m(\varepsilon)\}$, such that $x_m \to \infty$ and

$$\frac{b^2(x_m;\varepsilon)}{b^2(x_m;\lambda-\varepsilon)} \to 1, \quad \frac{b^2(x_m;\lambda+\varepsilon)}{b^2(x_m;\lambda-\varepsilon)} \to 0, \quad \frac{\Psi_1^2(\frac{\varepsilon}{x_m})}{b^2(x_m;\lambda-\varepsilon)} \to 0.$$

Because of the monotonicity of the functions $b^2(x,\varepsilon)$ and $\Psi_1^2(\varepsilon/x)$ in ε, none of the three ratios above increases in ε. Using this, one can easily show that the above conditions hold if and only if there exists a subsequence $\{n'\}$ of the positive integers, independent of ε, such that

$$(1.17) \quad \lim_{n'\to\infty} \frac{b^2(n';\varepsilon)}{b^2(n';\lambda-\varepsilon)} = 1, \qquad 0 < \varepsilon \leq \lambda/2,$$

$$(1.18) \quad \lim_{n'\to\infty} \frac{b^2(n';\lambda+\varepsilon)}{b^2(n';\lambda-\varepsilon)} = 0, \qquad 0 < \varepsilon \leq \lambda/2,$$

$$(1.19) \quad \lim_{n'\to\infty} \frac{\Psi_1^2(\varepsilon/n')}{b^2(n';\lambda-\varepsilon)} = 0, \qquad 0 < \varepsilon \leq \lambda/2.$$

The conditions (1.17)-(1.19) are obviously equivalent to

$$(1.20) \quad \lim_{n'\to\infty} \frac{b^2(n';\varepsilon) + b^2(n';\lambda+\varepsilon) + \Psi_1^2(\varepsilon/n')}{b^2(n';\lambda-\varepsilon)} = 1, \ 0 < \varepsilon \leq \lambda/2,$$

and the latter implies, of course, (1.16). Conversely, (1.16) implies (1.17)-(1.19), again by the monotonicity of the functions b^2 and Ψ_1^2 with respect to ε.

Some results, obtained in the proofs of our theorems, will be used to establish the following Corollary characterizing the two different ways of behaviour of the normalizing constants in (1.1) by comparing $A_{n'}$ and $a_2(n')$.

COROLLARY. *Suppose* (1.1). *If* $\lambda > 1$, *then there exist two positive constants* $\delta_1 < \delta_2$ *such that*

$$\delta_1 \leq \frac{a_2(n')}{A_{n'}} \leq \delta_2.$$

If $\lambda < 1$, *then* $\frac{a_2(n')}{A_{n'}} \to 0$ *as* $n' \to \infty$.

The proofs are in the next section. In Section 3, we illustrate the above results by a concrete construction of Ψ_2, that is, a concrete constraction of an infinitely divisible random variable in D_p(Poisson (λ)). This construction works for all $\lambda > 0$ and it will turn out that the case when $\lambda = 1$ is really a borderline case in the sense that both possibilities of the Corollary show up.

2. PROOFS

It is well known (see [4]) that for the Poisson law with mean $\lambda > 0$ the constant σ^2 in the representation (1.5) equals zero, $\theta = \lambda/2$, $L(x) \equiv 0$, $x > 0$, $R(x) = -\lambda$ for $0 < x < 1$ and $R(x) = 0$ for $x \geq 1$. Simple calculations show, that in this case the function difined by (1.6) is identically equal to zero and the one in (1.7), which we denote by Ψ_λ in the sequel, is given by

$$\Psi_\lambda(u) = \begin{cases} -1, & \text{if } 0 < u < \lambda, \\ 0, & \text{if } u \geq \lambda. \end{cases}$$

Furthermore we need the random variable $\overline{V}(0,\Psi_\lambda, 0)$ defined by (1.10). Easy calculations in (1.8) give

(2.1) $\overline{V}(0,\Psi_\lambda,0) = N(\lambda) - \lambda I\{\lambda > 1\} =_D Y_\lambda - \lambda I\{\lambda > 1\}$,

where, for simplicity, we put $N(\lambda) = N_2(\lambda)$.

Proof of Theorem 1. Let $\overline{V}_j(\Psi_1,\Psi_2,\sigma)$, $j = 1,2,\ldots$, be independent copies of $\overline{V}(\Psi_1,\Psi_2,\sigma)$ and suppose (1.1). This means, according to the remark following (1.10) that we have

$$\frac{1}{A_{n'}}\{\sum_{j=1}^{n'} \overline{V}_j(\Psi_1,\Psi_2,\sigma) - B_{n'}\} \to_D Y_\lambda, \quad n' \to \infty,$$

for some $A_{n'} > 0$ and $B_{n'}$. Let $\{\overline{V}_j^{(m)}\}_{j=1}^{\infty}$ be independent copies of $\{\overline{V}_j(\Psi_1,\Psi_2,\sigma)\}_{j=1}^{\infty}$, $m = 1,2,\ldots,r$, where $r \geq r_1(\lambda)$. Then

$$\frac{1}{A_{n'}}\{\sum_{j=1}^{n'} (\sum_{m=1}^{r} \overline{V}_j^{(m)}) - rB_{n'}\} \to_D Y_{r\lambda}, \; n' \to \infty.$$

Introducing

$$\overline{Z}_{(j-1)r+m} = \overline{V}_j^{(m)}, \quad 1 \leq m \leq r, \quad j = 1,2,\ldots$$

and $C^*_{n'} = rB_{n'} + r\lambda A_{n'}$ and using (2.1) we see that the above means

$$\frac{1}{A_{n'}}\{\sum_{j=1}^{rn'} \overline{Z}_j - C^*_{n'}\} \to_D \overline{V}(0,\Psi_{r\lambda},0).$$

We are in the situation of part (v) of Theorem 12 in [1], which necessarily implies

(2.2) $\frac{1}{A_{n'}} \Psi_1(\frac{s}{rn'}) \to 0$, if $s > 0$,

(2.3) $\frac{1}{A_{n'}} \Psi_2(\frac{s}{rn'}) \to \begin{cases} -1, & \text{if } 0 < s < r\lambda, \\ 0, & \text{if } s > r\lambda, \end{cases}$

as $n' \to \infty$, and

$$(2.4) \qquad \lim_{h\to\infty} \limsup_{n'\to\infty} \frac{a_1(rn';h)}{A_{n'}} = 0.$$

Also,

$$\limsup_{n'\to\infty} \frac{|\Psi_j(\frac{s}{rn'})|}{a_2(rn')} < \infty, \text{ for } s > 0,\ j = 1,2,$$

since otherwise we would have $a_2(rn')/A_{n'} \to 0$ as $n' \to \infty$, which by part (iv) of Theorem 12 in [1] is impossible because the limit in (2.3) is non-zero for $1 \le s < r\lambda$. Therefore, again by part (v) of the same Theorem 12, there exist an $\{n''\} \subset \{n'\}$ and a constant $0 < \alpha_r < \infty$ such that

$$\frac{a_2(rn'')}{A_{n'}} \to \frac{1}{\alpha_r} .$$

Putting this together with (2.2), (2.3) and (2.4), we obtain (1.11), (1.12) and (1.13) along $\{n''\}$ since $a_1(rn'';h) = \sqrt{r}\, a_1(n'';h/r)$ and, obviously,

$$\lim_{h\to\infty} \limsup_{n''\to\infty} \frac{a_1(n'';h/r)}{a_2(rn'')}$$

$$= \lim_{h\to\infty} \limsup_{n''\to\infty} \frac{a_1(n'';h)}{a_2(rn'')} .$$

This completes the proof of necessity.

Now we show sufficiency. Writing n_k' for the k-th term of the sequence $\{n'\}$ in (1.11), (1.12) and (1.13) and setting

$$\Psi_{jk}(\cdot) = \Psi_j(\cdot/n_k')/a_2(rn_k') := \Psi_j^{(n_k')}(\cdot)/a_2(rn_k'),$$

$j = 1,2,;\ k = 1,2,\ldots$, by the distributional equality (2.42) of [1] we get

$$(2.5)\quad \frac{1}{a_2(rn_k')}\{\sum_{j=1}^{n_k'} \nabla_j(\Psi_1,\Psi_2,\sigma) - \mu_{n_k'}(\Psi_1,\Psi_2)\}$$

$$=_D \nabla(\Psi_{1k},\Psi_{2k}, \frac{\sqrt{n_k'}\,\sigma}{a_2(rn_k')}),$$

where

$$\mu_n(\Psi_1,\Psi_2) = n\{\int_{1/n}^{1} u d\Psi_2(u) - \int_{1/n}^{1} u d\Psi_1(u)\}.$$

The general convergence theorem for infinitely divisible laws, that is, Theorem 11 in [1] states that under the conditions (1.11), (1.12) and (1.13),

$$\nabla(\Psi_{1k},\Psi_{2k}, \sqrt{n_k'}\,\sigma/a_2(rn_k')) \to_D \nabla(0,\alpha_r\Psi_\lambda,0)$$

as $k \to \infty$. By (2.5) and (2.1) we get

$$\frac{1}{a_2(rn_k')}\{\sum_{j=1}^{n_k'} \nabla_j(\Psi_1,\Psi_2,\sigma) - \mu_{n_k'}(\Psi_1,\Psi_2)\} \to \alpha_r(Y_\lambda - I\{\lambda > 1\}),$$

as $k \to \infty$, which proves sufficiency. □

The following four lemmas are required for the proof of Theorem 2.

LEMMA 1. *Let $\Lambda > 1$ be a fixed number and $\{n'\}$ be a subsequence such that*

$$(2.6)\quad \frac{1}{a_2(n')}\Psi_1(\frac{s}{n'}) \to 0,\ s > 0,$$

$$(2.7)\quad \frac{1}{a_2(n')}\Psi_2(\frac{s}{n'}) \to \begin{cases} -\alpha, & 0 < s < \Lambda, \\ 0, & s > \Lambda, \end{cases}$$

as $n' \to \infty$, where $\alpha > 0$ is some constant, and

$$(2.8)\quad \lim_{h\to\infty} \limsup_{n'\to\infty} \frac{a_1(n';h)}{a_2(n')} = 0.$$

Then, as $n' \to \infty$,

$$(2.9) \qquad \frac{b^2(n';h)}{b^2(n';1)} \to 1, \qquad 0 < h < \Lambda,$$

and

$$(2.10) \qquad \frac{b^2(n';h)}{b^2(n';1)} \to 0, \qquad h > \Lambda.$$

Proof. By (1.14) we have

$$(2.11) \qquad \frac{b^2(n';h)}{b^2(n';1)} = \frac{a_1^2(n';h)}{a_2^2(n';1)} + h\left\{\frac{\Psi_1^2(\frac{h}{n'})}{a_2^2(n';1)} + \frac{\Psi_2^2(\frac{h}{n'})}{a_2^2(n';1)}\right\}.$$

First we investigate the behaviour of the right-hand side of (2.11). Let $0 < h < s$ and consider

$$a_1^2(n;h) - a_1^2(n;s) = n\left\{\int_{h/n}^{s/n} \Psi_1^2(u)\,du + \int_{h/n}^{s/n} \Psi_2^2(u)\,du\right\}.$$

Then

$$n\left(\frac{s}{n} - \frac{h}{n}\right)\left\{\Psi_1^2\left(\frac{s}{n}\right) + \Psi_2^2\left(\frac{s}{n}\right)\right\} \le a_1^2(n;h) - a_1^2(n;s)$$

$$\le n\left(\frac{s}{n} - \frac{h}{n}\right)\left\{\Psi_1^2\left(\frac{h}{n}\right) + \Psi_2^2\left(\frac{h}{n}\right)\right\}$$

and hence

$$(2.12) \qquad (s-h)\left\{\frac{\Psi_1^2(\frac{s}{n})}{a_2^2(n;1)} + \frac{\Psi_2^2(\frac{s}{n})}{a_2^2(n;1)}\right\} \le \frac{a_1^2(n;h)}{a_2^2(n;1)} - \frac{a_1^2(n;s)}{a_2^2(n;1)}$$

$$\le (s-h)\left\{\frac{\Psi_1^2(\frac{h}{n})}{a_2^2(n;1)} + \frac{\Psi_2^2(\frac{h}{n})}{a_2^2(n;1)}\right\}.$$

Let $h > \Lambda$. By (1.14), (2.6) and (2.7) the right-hand side of (2.12) goes to zero as $n' \to \infty$ and this gives

$$0 \le \frac{a_1^2(n';h)}{a_2^2(n';1)} - \frac{a_1^2(n';s)}{a_2^2(n';1)} \to 0 \text{ as } n' \to \infty,\ \Lambda < h < s,$$

which implies

$$(2.13)\quad \limsup_{n'\to\infty} \frac{a_1^2(n';h)}{a_2^2(n';1)} = \limsup_{n'\to\infty} \frac{a_1^2(n';s)}{a_2^2(n';1)} =: \Delta^2(s).$$

Since $\Delta^2(s)$ does not increase, there exists $\lim_{s\to\infty} \Delta^2(s) =: \Delta \ge 0$.

Now (2.13) and (2.8) imply $\Delta = 0$ and this together with (2.6), (2.7) and (2.11) prove (2.10). To prove (2.9), put first $s = 1$, $0 < h < 1$ in (2.12). By (2.6) and (2.7) and by $\Lambda > 1$ the inequalities (2.12) give

$$\lim_{n'\to\infty} \left\{ \frac{a_1^2(n';h)}{a_2^2(n';1)} - \frac{a_1^2(n';1)}{a_2^2(n';1)} \right\} = (1-h)\alpha^2,$$

while by putting $h = 1$, $1 < s < \Lambda$ they give

$$\lim_{n'\to\infty} \left\{ \frac{a_1^2(n';1)}{a_2^2(n';1)} - \frac{a_1^2(n';s)}{a_2^2(n';1)} \right\} = (s-1)\alpha^2.$$

These two limiting equalities together with

$$\frac{a_1^2(n';1)}{a_2^2(n';1)} = 1 - \left\{ \frac{\Psi_1^2(\frac{1}{n'})}{a_2^2(n';1)} + \frac{\Psi_2^2(\frac{1}{n'})}{a_2^2(n';1)} \right\} \to 1 - \alpha^2,$$

as $n' \to \infty$, imply that

$$(2.14)\quad \frac{a_1^2(n';h)}{a_2^2(n';1)} \to 1 - h\alpha^2, \text{ as } n' \to \infty,\quad 0 < h < \Lambda.$$

Now by (2.6), (2.7) and (2.14) used in (2.11) we easily get (2.9).

LEMMA 2. *Let* $\Lambda > 1$ *be a fixed number and suppose* (2.9), (2.10) *and that*

(2.15) $$\frac{\Psi_1(\frac{1}{n'})}{b(n';1)} \to 0, \text{ as } n' \to \infty .$$

Then there exist a subsequence $\{n''\} \subset \{n'\}$ *and a constant* $\alpha > 0$ *such that* (2.6), (2.7) *and* (2.8) *are satisfied with* n' *replaced by* n''.

Proof. For $h > \Lambda$, (2.10) and (2.11) give

$$\frac{a_1^2(n';h)}{a_2^2(n';1)} \to 0, \text{ as } n' \to \infty$$

which implies of course (2.8) along the original $\{n'\}$. Furthermore, the definition of b^2 and (1.14) easily imply that for $h < s$,

$$\frac{a_1^2(n;h)}{a_2^2(n;1)} - \frac{a_1^2(n;s)}{a_2^2(n;1)} = \frac{b^2(n;h)}{b^2(n;1)} - \frac{b^2(n;s)}{b^2(n;1)}$$

$$+ s\{ \frac{\Psi_1^2(\frac{s}{n})}{a_2^2(n;1)} + \frac{\Psi_2^2(\frac{s}{n})}{a_2^2(n;1)}\} - h\{ \frac{\Psi_1^2(\frac{h}{n})}{a_2^2(n;1)} + \frac{\Psi_2^2(\frac{h}{n})}{a_2^2(n;1)} \} .$$

We substitute the right-hand side of the above equality into (2.12) and get

(2.16) $$h\{ \frac{\Psi_1^2(\frac{h}{n})}{a_2^2(n;1)} - \frac{\Psi_1^2(\frac{s}{n})}{a_2^2(n;1)}\} + h\{ \frac{\Psi_2^2(\frac{h}{n})}{a_2^2(n;1)} - \frac{\Psi_2^2(\frac{s}{n})}{a_2^2(n;1)}\}$$

$$\le \frac{b^2(n;h)}{b^2(n;1)} - \frac{b^2(n;s)}{b^2(n;1)}$$

$$\le s\{ \frac{\Psi_1^2(\frac{h}{n})}{a_2^2(n;1)} - \frac{\Psi_1^2(\frac{s}{n})}{a_2^2(n;1)}\} + s\{ \frac{\Psi_2^2(\frac{h}{n})}{a_2^2(n;1)} - \frac{\Psi_2^2(\frac{s}{n})}{a_2^2(n;1)}\} .$$

By putting $0 < h < 1$, $s = 1$ into (2.16), and by (2.9) we get

$$0 \le \frac{\Psi_1^2(\frac{h}{n'})}{a_2^2(n';1)} - \frac{\Psi_1^2(\frac{1}{n'})}{a_2^2(n';1)} \to 0, \text{ as } n' \to \infty,$$

for $0 < h < 1$. From (2.15) and this (2.6) follows for $0 < s < 1$. Since for $s > 1$ we have by (2.15)

$$\frac{\Psi_1^2(\frac{s}{n'})}{a_2^2(n';1)} \le \frac{\Psi_1^2(\frac{1}{n'})}{a_2^2(n';1)} \to 0, \text{ as } n' \to \infty,$$

(2.6) is true for all $s > 0$. When $s = 1$ and $0 < h < 1$ in (2.16), then by (2.9) we obtain

$$(2.17) \qquad 0 \le \frac{\Psi_2^2(\frac{h}{n'})}{a_2^2(n';1)} - \frac{\Psi_2^2(\frac{1}{n'})}{a_2^2(n';1)} \to 0, \text{ as } n' \to \infty .$$

Since $\frac{\Psi_2^2(\frac{1}{n'})}{a_2^2(n';1)} \le 1$, we can find an $\{n''\} \subset \{n'\}$ such that

$$\frac{\Psi_2^2(\frac{1}{n''})}{a_2^2(n'';1)} \to \alpha^2 \ge 0, \text{ as } n'' \to \infty,$$

which and (2.17) give

$$\frac{\Psi_2^2(\frac{h}{n''})}{a_2^2(n'';1)} \to \alpha^2, \text{ as } n'' \to \infty, \text{ for } 0 < h < 1.$$

When putting $h = 1$ and $1 < s < \Lambda$ in (2.16) and using (2.9) we obtain that this relation holds true for $1 < h < \Lambda$ as well. Using this and the relation

$$\frac{\Psi_2^2(\frac{h}{n''})}{a_2^2(n'';1)} \le \frac{b^2(n'';h)}{b^2(n'';1)} \to 0, \text{ as } n'' \to \infty,$$

which according to (2.11) and (2.10) is true for $h > \Lambda$, we get

$$(2.18) \qquad \frac{\Psi_2^2(\frac{h}{n''})}{a_2^2(n'';1)} \to \begin{cases} \alpha^2, & \text{if } 0 < h < \Lambda , \\ 0 , & \text{if } h > \Lambda , \end{cases}$$

as $n'' \to \infty$. We claim that $\alpha^2 > 0$. Suppose not. Then

$$\frac{\Psi_2^2(\frac{1}{n''})}{a_2^2(n'';1)} \to 0, \text{ as } n'' \to \infty$$

and together with (2.15) this leads to

$$\frac{a_1^2(n'';1)}{a_2^2(n'';1)} = 1 - \frac{\Psi_1^2(\frac{1}{n''})}{a_2^2(n'';1)} - \frac{\Psi_2^2(\frac{1}{n''})}{a_2^2(n'';1)} \to 1, \text{as } n'' \to \infty .$$

The condition above, the already proved (2.6) and (2.18) with $\alpha = 0$ show that we are in the case (i) of Theorem 12 of [1]. Then part (v) of the theorem gives that along $\{n''\}$ we should have

$$\lim_{h\to\infty} \limsup_{n''\to\infty} \frac{a_1^2(n'';h)}{a_2^2(n'';1)} = 1$$

which contradicts the already proved (2.8). Hence $\alpha^2 > 0$ and this completes the proof. □

LEMMA 3. *Suppose* (2.9), (2.10) *and* (2.15) *are satisfied along* $\{n'\}$ *and* $\Lambda > 1$. *Then for every* $0 < \varepsilon \le \Lambda/2$,

$$(2.19) \qquad \frac{b^2(n';\varepsilon)}{b^2(n';\Lambda-\varepsilon)} \to 1,$$

$$(2.20) \qquad \frac{b^2(n';\Lambda+\varepsilon)}{b^2(n';\Lambda-\varepsilon)} \to 0,$$

$$(2.21) \qquad \frac{\Psi_1^2(\frac{\varepsilon}{n'})}{b^2(n';\Lambda-\varepsilon)} \to 0 ,$$

as $n' \to \infty$.

Proof. Take any ε, $0 < \varepsilon \le \Lambda/2$. By (2.9),

$$\frac{b^2(n';\varepsilon)}{b^2(n';\Lambda-\varepsilon)} = \frac{b^2(n';\varepsilon)}{b^2(n';1)} \left[\frac{b^2(n';\Lambda-\varepsilon)}{b^2(n';1)}\right]^{-1} \to 1, \; n' \to \infty,$$

which is (2.19). Further by (2.9) and (2.10) for the ratio in (2.20) we have

$$\frac{b^2(n';\Lambda+\varepsilon)}{b^2(n';\Lambda-\varepsilon)} = \frac{b^2(n';\Lambda+\varepsilon)}{b^2(n';1)} \left[\frac{b^2(n';\Lambda-\varepsilon)}{b^2(n';1)}\right]^{-1} \to 0, \; n' \to \infty,$$

and finally (2.15) and (2.9) give

$$\frac{\psi_1^2(\frac{\varepsilon}{n'})}{b^2(n';\Lambda-\varepsilon)} = \frac{\psi_1^2(\frac{\varepsilon}{n'})}{b^2(n';1)} \left[\frac{b^2(n';\Lambda-\varepsilon)}{b^2(n';1)}\right]^{-1} \to 0, \quad n' \to \infty,$$

which is (2.21).

LEMMA 4. *Suppose that* (2.19), (2.20) *and* (2.21) *hold true along some subsequence* $\{n'\}$ *with* $\Lambda > 2$. *Then* (2.9), (2.10) *and* (2.15) *hold true along the same subsequence and with the same* Λ.

Proof. The condition $\Lambda > 2$ implies that for $0 < \varepsilon \le \Lambda/2$, $\Lambda - \varepsilon > 1$. Then by (2.19),

$$1 \le \frac{b^2(n';\varepsilon)}{b^2(n';1)} \le \frac{b^2(n';\varepsilon)}{b^2(n';\Lambda-\varepsilon)} \to 1, \; n' \to \infty, \quad 0 < \varepsilon \le 1,$$

and

$$1 \le \frac{b^2(n';1)}{b^2(n';\varepsilon)} \le \frac{b^2(n';1)}{b^2(n';\Lambda-1)} \to 1, \; n' \to \infty, \; 1 < \varepsilon \le \Lambda/2,$$

which together give (2.9).

Furthermore, by (2.20) and (2.19) for $0 < \varepsilon \le \Lambda/2$ we have

$$\frac{b^2(n';\Lambda+\varepsilon)}{b^2(n';1)} = \frac{b^2(n';\Lambda+\varepsilon)}{b^2(n';\Lambda-\varepsilon)} \quad \frac{b^2(n';\Lambda-\varepsilon)}{b^2(n';1)} \to 0, \; n' \to \infty,$$

and since for every $h > \Lambda$ there exists a small ε, such that $h > \Lambda+\varepsilon$ and then

$$\frac{b^2(n';h)}{b^2(n';1)} \le \frac{b^2(n';\ \Lambda+\varepsilon)}{b^2(n';1)} \to 0 \ ,$$

we have (2.10) also satisfied. Finally we use (2.21) with $\varepsilon = 1$ and get

$$\frac{\Psi_1^2(\frac{1}{n'})}{b^2(n';1)} \le \frac{\Psi_1^2(\frac{1}{n'})}{b^2(n';\ \Lambda-1)} \to 0,\ n' \to \infty,$$

which is (2.15). □

Proof of Theorem 2. First we prove necessity. Assume (1.11), (1.12) and (1.13) for some $\lambda > 0$, a fixed $r \ge r_1(\lambda) \ge 1$ and some $\{n'\}$. Then we have

$$(2.22)\qquad \frac{1}{a_2(rn')}\,\Psi_1(\frac{s}{rn'}) \to 0, \quad s > 0,$$

$$(2.23)\qquad \frac{1}{a_2(rn')}\,\Psi_2(\frac{s}{rn'}) \to \begin{cases} -\alpha_r, & 0 < s < r\lambda, \\ 0\ , & s > r\lambda, \end{cases}$$

as $n' \to \infty$, where $\alpha_r > 0$ is some constant, and

$$(2.24)\qquad \lim_{h\to\infty}\ \limsup_{n'\to\infty} \frac{a_1(rn';h)}{a_2(rn')} = 0.$$

By Lemma 1, (2.22), (2.23) and (2.24) give

$$(2.25)\qquad \frac{b^2(rn';h)}{b^2(rn';1)} \to \begin{cases} 1, & 0 < h < r\lambda, \\ 0, & h > r\lambda, \end{cases}$$

and, as a special case of (2.22), we have also

$$(2.26)\qquad \frac{\Psi_1(\frac{1}{rn'})}{a_2(rn')} \to 0\ ,$$

as $n' \to \infty$. By Lemma 3, applied along $\{rn'\}$ and with $\Lambda = r\lambda$, we obtain

$$(2.27) \qquad \lim_{n'\to\infty} \frac{b^2(rn';h) + b^2(rn'; r\lambda+h) + r\Psi_1^2(\frac{h}{rn'})}{b^2(rn'; r\lambda-h)} = 1$$

for $0 < h < r\lambda/2$. Since

$$b^2(rn';h) = r\, a_1^2(n';h/r) + h\{\Psi_1^2(\frac{h}{rn'}) + \Psi_2^2(\frac{h}{rn'})\}$$

$$= r\{a_1^2(n';\frac{h}{r}) + \frac{h}{r}(\Psi_1^2(\frac{h/r}{n'}) + \Psi_2^2(\frac{h/r}{n'}))\}$$

$$= rb^2(n'; h/r),$$

by putting $\varepsilon = h/r$ in (2.27) we obtain (1.20) and hence (1.16).

Now assume (1.16). Then of course we have (1.17), (1.18) and (1.19) along some subsequence $\{n'\}$. Take $r_2 = r_2(\lambda) = \min\{r: r \text{ positive integer}, r\lambda > 2\}$ and write (1.17), (1.18) and (1.19) in the form of (2.27) with $r = r_2$ and $h = r_2\varepsilon$. By Lemma 4 the latter implies (2.25) and (2.26) with $r = r_2$. By Lemma 2, used for the subsequence $\{r_2n'\}$ and $\Lambda = r_2\lambda > 2$ we can find an $\{n''\} \subset \{n'\}$ such that (2.6), (2.7) and (2.8) are satisfied along $\{r_2n''\}$ for $\Lambda = r_2\lambda$ and some $\alpha > 0$. Further we substitute $t = s/r_2$ and get (1.11), (1.12) and (1.13) with n' replaced by n''. Then, as shown in the sufficiency proof of Theorem 1, $(\Psi_1, \Psi_2, \sigma) \in D_p(\text{Poisson } (\lambda))$. □

Proof of the Corollary. Suppose (1.1). Following the necessity proof of Theorem 1, one can see, that if $r \geq r_1(\lambda)$ then for every $\{n''\} \subset \{n'\}$ there exists an $\{n'''\} \subset \{n''\}$ such that we have (2.6), (2.7) and (2.8) along $\{rn'''\}$ with $\Lambda = r\lambda$ and, as $n''' \to \infty$,

$$\frac{a_2(rn''')}{A_{n'''}} \to \delta,$$

where $0 < \delta < \infty$. This is nothing but

$$(2.28)\qquad \delta_1 \le \frac{a_2(rn')}{A_{n'}} \le \delta_2,$$

where $0 < \delta_1 = \delta_1(r) < \delta_2 = \delta_2(r) < \infty$. If $\lambda > 1$, then $r_1(\lambda) = 1$ and we get the first statement of the Corollary by putting $r = 1$ in (2.28). Let $\lambda < 1$ and $r > \max(r_1(\lambda), \frac{1}{1-\lambda})$. Then

$$\frac{a_2^2(n')}{A_{n'}^2} = \frac{b^2(n';1)}{A_{n'}^2} = \frac{b^2(n';1)}{b^2(rn';1)} \quad \frac{a_2^2(rn')}{A_{n'}^2} \le \delta_2^2(r) \quad \frac{b^2(n';1)}{rb^2(n';1/r)}$$

$$= \frac{\delta_2^2(r)}{r} \quad \frac{b^2(n';1)}{b^2(n';\lambda-1/r)} \quad \frac{b^2(n';\lambda-1/r)}{b^2(n';1/r)} \to 0, \text{ as } n' \to \infty,$$

since, using first Lemma 1 along $\{rn'''\}$ and with $\Lambda = r\lambda$ and then Lemma 3 along $\{rn'''\}$ and with $\Lambda=r\lambda$ and $\varepsilon=1$ and noting that $r > \Lambda+\varepsilon$

$$\frac{b^2(n''';1)}{b^2(n''';\lambda-1/r)} \to 0, \quad \frac{b^2(n''';\lambda-1/r)}{b^2(n''';1/r)} \to 1, \text{ as } n''' \to \infty,$$

and since $\{n''\}$ was arbitrary. □

3. A CONSTRUCTION

Denote

$$t_j = t_j(c) = c\,2^{-2^j}, \; j = 0,1,2,\ldots,$$

where c is a number, $0 < c \le 1$, and

$$b_j = b_j(c) = (t_{j-1}-t_j)^{-1} = \frac{2^{2^{j-1}}}{c(1-2^{-2^{j-1}})}, \; j = 0,1,2,\ldots$$

Introduce the function

$$\Psi(s) = \begin{cases} -b_k, & t_k \le s < t_{k-1}, \quad k = 1,2,\ldots, \\ 0, & s \ge t_o, \end{cases}$$

and the corresponding infinitely divisible random variable $\overline{V}(0,\Psi,0)$, given by (1.10) and (1.8). Since $t_o = c/2 < 1$, simple calculations in (1.8) show that

$$\overline{V}(0,\Psi,0) = \int_0^{t_o} N(s)d\Psi(s) = \sum_{k=0}^{\infty} N(t_k)(b_{k+1} - b_k),$$

where $b_o = 0$.

Fix $\lambda > 0$. We will illustrate the results of Theorem 1 and of the corollary by choosing $X = \overline{V}(0,\Psi,0)$, $\{n'\} = \{n_k\}_{k=1}^{\infty}$, where

(3.1) $$n_k = \left\lceil \frac{\lambda}{t_k} \right\rceil = \min\{l : l \text{ integer}, \ l \ge \frac{\lambda}{t_k}\}, \quad k = 0,1,2,\ldots,$$

and $A_{n_k} = b_{k+1}$, $k = 0,1,2,\ldots,$

and discuss a bit the situation when n_k above is replaced by m_k, where

(3.2) $$m_k = \left\lfloor \frac{\lambda}{t_k} \right\rfloor = \max\{l : l \text{ integer}, \ l \le \frac{\lambda}{t_k}\}, \quad k = 0,1,2,\ldots$$

Let n_k be defined by (3.1). Then by elementary considerations we obtain that for all k large enough,

(3.3) $$\begin{aligned} t_{k+1} &< \frac{s}{n_k} < t_k, && \text{if } s < \lambda, \\ t_{k+1} &< \frac{\lambda}{n_k} < t_k, && \text{if } \left\lceil \frac{\lambda}{t_k} \right\rceil > \frac{\lambda}{t_k}, \\ t_k &= \frac{\lambda}{n_k} < t_{k-1}, && \text{if } \left\lceil \frac{\lambda}{t_k} \right\rceil = \frac{\lambda}{t_k}, \\ t_k &< \frac{s}{n_k} < t_{k-1}, && \text{if } s > \lambda. \end{aligned}$$

These together with the asymptotic equality

$$(3.4)\qquad b_k^2 \sim (1/c^2)2^{2^k}$$

imply that if $s \neq \lambda$,

$$(3.5)\qquad \frac{\Psi(\frac{s}{n_k})}{A_{n_k}} \to \Psi_\lambda(s) = \begin{cases} -1, & \text{if } s < \lambda, \\ 0, & \text{if } s \geq \lambda, \end{cases}$$

as $k \to \infty$. Further we want to know the behaviour of the ratios $a_1(n_k;h)/a_2(rn_k)$ and $a_2(n_k)/A_{n_k}$. For this we first analyse the function

$$a_1^2(n_k;\, h) = n_k \int_{h/n_k}^{t_o} \Psi^2(s)ds.$$

Relations (3.3) show that when $h < \lambda$ or $h = \lambda$ but $\left\lceil \frac{\lambda}{t_k} \right\rceil > \frac{\lambda}{t_k}$ then

$$a_1^2(n_k;\, h) = n_k \{ \sum_{j=1}^{k} (t_{j-1}-t_j)b_j^2 + (t_k - \frac{h}{n_k})b_{k+1}^2 \}.$$

By simple calculations this implies that when $k \to \infty$,

$$(3.6)\qquad a_1^2(n_k;\, h) \sim \frac{\lambda}{c^2}(1 - \frac{h}{\lambda})2^{2^{k+1}}, \text{ if } h < \lambda,$$

and

$$(3.7)\qquad \begin{aligned} a_1^2(n_k;\lambda) \sim \frac{\lambda}{c^2} 2^{2^k} \{ \sum_{j=1}^{k} 2^{2^{j-1}} \left[1-2^{-2^{j-1}}\right]^{-1} + \\ + \left[1 - \frac{\lambda}{c} 2^{2^k} (\left\lceil \frac{\lambda}{c} 2^{2^k} \right\rceil)^{-1}\right] 2^{2^k} \}, \text{ if } \left\lceil \frac{\lambda}{t_k} \right\rceil > \frac{\lambda}{t_k}. \end{aligned}$$

Again by (3.3) when $h > \lambda$ or $h = \lambda$ but $\left\lceil \frac{\lambda}{t_k} \right\rceil = \frac{\lambda}{t_k}$ we obtain

$$a_1^2(n_k;\, h) = n_k \{ \sum_{j=1}^{k-1} (t_{j-1}-t_j)b_j^2 + (t_{k-1} - \frac{h}{n_k})b_k^2 \}$$

and this implies that when $k \to \infty$,

$$(3.8)\qquad a_1^2(n_k;\, h) \sim \frac{\lambda}{c^2} 2^{2^k} 2^{2^{k-1}}, \quad \text{if } h > \lambda,$$

and

$$(3.9)\qquad a_1^2(n_k;\lambda) \sim \frac{\lambda}{c^2}\, 2^{2^k} \sum_{j=1}^{k} 2^{2^{j-1}}\left[1-2^{-2^{j-1}}\right]^{-1}, \text{ if } \left\lceil\frac{\lambda}{t_k}\right\rceil = \frac{\lambda}{t_k}\,.$$

Now the relation $a_2^2(n_k) = a_1^2(n_k;1) + \Psi^2(\frac{1}{n_k})$ and also (3.6)-(3.9) show that we have to distinguish four cases.

Case 1: $\lambda > 1$. By (3.3), $\Psi^2(\frac{1}{n_k}) = b_{k+1}^2$ and then by (3.6) and (3.4) used with $h = 1$ we get

$$(3.10)\qquad a_2^2(n_k) \sim \frac{\lambda}{c^2}\, 2^{2^{k+1}} \sim \frac{\lambda}{\lambda-1}\, a_1^2(n_k),\ k \to \infty\,.$$

From the above and using consecutively (3.8) and (3.4) we obtain that when $k \to \infty$,

$$\frac{a_1^2(n_k;h)}{a_2^2(n_k)} \to 0, \text{ for every } h > \lambda,$$

and

$$\frac{a_2^2(n_k)}{A_{n_k}^2} \longrightarrow \lambda.$$

These relations together with (3.5) illustrate Theorem 1 and the first case of the corollary.

Case 2: $\lambda < 1$. Since for every $r \geq r_1(\lambda)$ we have $1/r < \lambda$ and since

$$a_2^2(rn_k) = r\, a_2^2(n_k;\ 1/r) = r\{a_1^2(n_k;\ 1/r) + \Psi^2(\tfrac{1/r}{n_k})\},$$

by using (3.6) with $h = 1/r$ and (3.3) with $s = 1/r$ we obtain

$$a_2^2(rn_k) \sim \frac{r(\lambda+1)-1}{c^2}\, 2^{2^{k+1}},\ k \to \infty.$$

Then (3.8) implies that when $k \to \infty$,

$$(3.11)\qquad \frac{a_1^2(n_k;\ h)}{a_2^2(rn_k)} \to 0, \text{ for every } h > \lambda.$$

Furthermore, in this case by (3.3) $\Psi^2(\frac{1}{n_k}) = b_k^2$ and then by (3.8) and (3.4),

$$(3.12)\qquad a_2^2(n_k) \sim a_1^2(n_k) \sim \frac{\lambda}{c^2}\, 2^{2^k}\, 2^{2^{k-1}}, \quad k \to \infty,$$

which implies, of course, that

$$\frac{a_2^2(n_k)}{A_{n_k}^2} \to 0, \quad k \to \infty.$$

This, (3.5) and (3.11) together illustrate Theorem 1 and the second case of the corollary.

Case 3: $\lambda = 1$ *and we choose* $0 < c < 1$ *so that* $1/c$ *is not an integer*. By (3.6) and (3.3) used with $h = 1/r < 1$ and $s = 1/r$ correspondingly, where $r \geq r_1(1) = 2$, we get

$$a_2^2(rn_k) \sim \frac{2r-1}{c^2}\, 2^{2^{k+1}}, \quad k \to \infty.$$

This and (3.8) imply that when $k \to \infty$

$$(3.13)\qquad \frac{a_1^2(n_k;h)}{a_2^2(n_k)} \to 0, \text{ for every } h > 1.$$

Furthermore, by (3.3), $\Psi^2(\frac{1}{n_k}) = b_{k+1}^2$ and since by (3.7) and (3.4),

$$(3.14)\qquad \frac{a_1^2(n_k)}{b_{k+1}^2} \longrightarrow 0, \quad k \to \infty,$$

then in this case

$$(3.15)\qquad a_2^2(n_k) \sim b_{k+1}^2 = A_{n_k}^2, \quad k \to \infty.$$

It is seen now by (3.13) and (3.15) that we are in the situation of Case 1. (The present case also illustrates part (iii) of Theorem 12 in [1].)

Case 4: $\lambda = 1$ *and* $c = 1$. Relations (3.6) and (3.3) used with $h = 1/r < 1$ and $s = 1/r$ give

$$a_2^2(rn_k) \sim (r-1)2^{2^{k+1}}, \quad k \to \infty,$$

and then this and (3.8) imply (3.13). By (3.3), $\Psi^2(\frac{1}{n_k}) = b_k^2$ and by (3.4) and (3.9) used with $\lambda = 1$ we get

$$\frac{\Psi^2(\frac{1}{n_k})}{a_1^2(n_k)} \to 0, \quad k \to \infty,$$

which means that

$$(3.16) \qquad a_2^2(n_k) \sim a_1^2(n_k), \quad k \to \infty.$$

Using again (3.9) and (3.4) we obtain

$$\frac{a_2^2(n_k)}{A_{n_k}^2} \to 0, \quad k \to \infty.$$

The latter and (3.13) show that we are in the situation of Case 2.

It is perhaps interesting to remark the following. As we see, the role of the "natural" normalizing constant for our problem is taken by $a_2(n)$, while in the general case, that is, when the limit in (1.1) is an arbitrary (infinitely divisible) random variable this role may be taken either by $a_1(n)$ or by $a_2(n)$, as Theorem 12 of [1] shows. Relations (3.10), (3.12) and (3.16) show, that in cases 1, 2 and 4 of the above construction $a_1(n_k) \asymp a_2(n_k)$, $k \to \infty$, while in Case 4 $a_1(n_k)/a_2(n_k) \to 0$ and $a_2(n_k) \sim A_{n_k}$, $k \to \infty$, as (3.14) and (3.15) show. The situation changes if we replace n_k by m_k of (3.2). This second construction illustrates the same results in the same way, but in all four cases, that is, for every $\lambda > 0$, we have $a_1(m_k) \asymp a_2(m_k)$.

REFERENCES

1. CSÖRGŐ, S. A probabilistic approach to domains of partial attraction. Adv. in Appl. Math. To appear.
2. CSÖRGŐ, S., DODUNEKOVA, R. The domain of partial attraction of a Poisson law. 1989. Submitted.
3. CSÖRGŐ, S., HAEUSLER E. and MASON, D.M. A probabilistic approach to the asymptotic distribution of sums of independent, identically distributed random variables. Adv. in Appl. Math., 9, 1988, 259-233.
4. GNEDENKO, B.V. and KOLMOGOROV, A.N. Limit distributions for sums of independent random variables. Addison-Wesley, Reading. Massachusetts, 1954.
5. GROSHEV, A.V. The domain of attraction of the Poisson law. Izvestija Akad. Nauk USSR, Ser. Mat. 5, 1941, 165-172 [Russian]

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