(#2014)

THE QUANTILE-TRANSFORM APPROACH TO THE ASYMPTOTIC DISTRIBUTION OF MODULUS TRIMMED SUMS

SÁNDOR CSÖRGŐ * ERICH HAEUSLER DAVID M. MASON †

1. Introduction and statement of results. Let $X, X_1, X_2, \ldots,$ be independent random variables with a common non-degenerate distribution function F which is *symmetric about zero.* For $1 \le j \le n$, set

$$m_n(j) = \#\{1 \le i \le n : |X_i| > |X_j| \text{ or } (|X_i| = |X_j| \text{ and } i \le j)\},$$

and let ${}^{(k)}X_n = X_j$ if $m_n(j) = k,\ 1 \le k \le n$. Thus ${}^{(k)}X_n$ is the k^{th} largest random variable in absolute value among $X_1, \ldots, X_n$, with ties broken according to the order in which the sample occurs. Consider the 'modulus' trimmed sums defined for $0 \le k \le n-1$ as

$${}^{(k)}S_n = {}^{(k+1)}X_n + \ldots + {}^{(n)}X_n.$$

Following a preliminary investigation by Pruitt [7], Griffin and Pruitt [4] undertook a detailed study of the asymptotic distribution of ${}^{(k_n)}S_n$, when $\{k_n\}$ is a sequence of integers such that $0 \le k_n \le n-1$ for $n \ge 1$ and both

$$k_n \to \infty \text{ and } k_n/n \to 0 \text{ as } n \to \infty. \tag{1.1}$$

The aim of the present paper is to illuminate their main results by means of our quantile-transform–weak-approximation approach.

*Partially supported by the Hungarian National Foundation for Scientific Research, Grants 1808/86 and 457/88

†Partially supported by a Fulbright Grant and NSF Grant #DMS-8803209.

Let

$$G(x) = P\{|X| \leq x\},\ -\infty < x < \infty,$$

be the distribution function of $|X|$, with corresponding quantile function

$$K(s) = \inf\{x : G(x) \geq s\},\ 0 < s \leq 1,\ K(0) = K(0+).$$

Moreover, for each integer $n \geq 1$ and $-\infty < x < \infty$ set

$$\tau_n^2(x) = n \int_0^{u_n(x)} K^2(u)du,$$

where $u_n(x) = \{(n - k_n + x k_n^{\frac{1}{2}})n^{-1} \vee 0\} \wedge 1$ with $\vee$ and $\wedge$ standing for maximum and minimum.

The following theorem yields the direct halves of Theorems 3.2, 3.4, 3.5, 3.7, 3.9 and 3.10 of Griffin and Pruitt [4] under the cleaner forms of their conditions, which correspond to those given in their paper when the underlying distribution function is continuous. In what follows, the symbols $=_{\mathcal{D}}$ and $\rightarrow_{\mathcal{D}}$ will mean equality and convergence in distribution, respectively.

THEOREM. *Let $0 \leq k_n \leq n-1$ for $n \geq 1$ be a sequence of positive integers satisfying (1.1). Assume that for a non-degenerate distribution function F, symmetric about zero, and a subsequence $\{n'\} \subset \{n\}$ there exist $A_{n'} > 0$ and $B_{n'}$ such that for some $-\infty < C < \infty$,*

$$-B_{n'}/A_{n'} \rightarrow C \quad as \quad n' \rightarrow \infty \tag{1.2}$$

and for all $-\infty < x < \infty$

$$v_{n'}^2(x) := \tau_{n'}^2(x)/A_{n'}^2 \rightarrow v^2(x) \quad as \quad n' \rightarrow \infty, \tag{1.3}$$

for a (necessarily non-decreasing convex) function v^2 which is not identically equal to zero. Then

$$A_{n'}^{-1}\{^{(k_{n'})}S_{n'} - B_{n'}\} \rightarrow_{\mathcal{D}} V \quad as \quad n' \rightarrow \infty, \tag{1.4}$$

where the limiting random variable V is the variance mixture of normals with characteristic function

$$E\{\exp(itV)\} = \exp(iCt)\int_{-\infty}^{\infty} \exp(-\frac{v^2(x)t^2}{2})d\Phi(x),\ -\infty < t < \infty,$$

where Φ is the standard normal distribution function.

The proof, postponed until the next section, is based on a weighted approximation to a 'signed' empirical process, which is likely to be of separate interest. See the Proposition in Section 2.

From our theorem it is easy to infer that a sufficient condition for the asymptotic normality of ${}^{(k_{n'})}S_{n'}$ is that for all $-\infty < x < \infty$,

$$\tau_{n'}^2(x)/\tau_{n'}^2(0) \to 1 \quad \text{as} \quad n' \to \infty, \tag{1.5}$$

implying that

$${}^{(k_{n'})}S_{n'}/\tau_{n'}(0) \to_{\mathcal{D}} V,$$

where V has the characteristic function for $-\infty < t < \infty$,

$$E\{\exp(itV)\} = \int_{-\infty}^{\infty} \exp(-\frac{t^2}{2})d\Phi(x) = e^{-\frac{t^2}{2}},$$

saying that V is standard normal. (Arguing as in Griffin and Pruitt [4], it can be shown that (1.5) is also necessary for the asymptotic non-degenerate normality of ${}^{(k_{n'})}S_{n'}$.)

It is elementary to show that (1.5) is equivalent to

$$k_{n'}^{1/2}K^2(u_{n'}(x))/\tau_{n'}^2(0) \to 0 \quad \text{as} \quad n' \to \infty \tag{1.6}$$

for all $-\infty < x < \infty$. From this form of the normality condition it can be seen that stochastic compactness of the whole sums S_n, which in the present symmetric case is equivalent to

$$\limsup_{s\downarrow 0} \; sK^2(1-s) \Big/ \int_0^{1-s} K^2(u)du < \infty, \tag{1.7}$$

implies that ${}^{(k_n)}S_n/\tau_n(0)$ converges in distribution to a standard normal random variable for *all* sequences k_n satisfying (1.1) as $n \to \infty$. (Refer to Corollary 10 and (3.13) in [1].) That stochastic compactness of the whole sums S_n implies asymptotic normality of ${}^{(k_n)}S_n$ for all sequences k_n as in (1.1) was first pointed out by Pruitt [7].

2. Proof. In the proof of the theorem we work on a probability space that carries a sequence $U, U_1, U_2, \ldots,$ of independent random variables uniformly distributed on (0,1) and, independent of this sequence, a sequence $s, s_1, s_2, \ldots,$ of independent and identically distributed random signs, that is, $P\{s = 1\} = P\{s = -1\} = \frac{1}{2}$. Since $K(U) =_{\mathcal{D}} |X|$, it is elementary to show using symmetry that

$$(X, |X|) =_{\mathcal{D}} (sK(U), K(U)),$$

from which we have immediately that

$$\{(X_i, |X_i|) : i \geq 1\} =_{\mathcal{D}} \{(s_i K(U_i), K(U_i)) : \ i \geq 1\}.$$

For each $n \geq 1$, let $U_{1,n} \leq \ldots \leq U_{n,n}$ denote the order statistics of $U_1, \ldots, U_n$. Let $D_{1,n}, \ldots, D_{n,n}$ denote the antiranks of $U_1, \ldots, U_n$, i.e. $U_{D_{i,n}} = U_{i,n}$ for $1 \leq i \leq n$. Then from the above distributional equality we easily obtain that for each fixed $n \geq 1$,

$$({}^{(n)}X_n, \ldots, {}^{(1)}X_n) =_{\mathcal{D}} (s_{D_{1,n}} K(U_{1,n}), \ldots, s_{D_{n,n}} K(U_{n,n})),$$

so that

$${}^{(k_n)}S_n =_{\mathcal{D}} \sum_{i=1}^{n-k_n} s_{D_{i,n}} K(U_{i,n}).$$

For each $n \geq 1$, we define the 'signed' empirical process

$$H_n(t) = n^{-1} \sum_{i=1}^{n} s_i I(\{U_i \leq t\}), \ 0 \leq t \leq 1,$$

where $I(A)$ denotes the indicator of a set A. Notice that almost surely for $n \geq 1$,

$$H_n(t) = n^{-1} \sum_{i=1}^{n} s_{D_{i,n}} I(\{U_{i,n} \leq t\}), \ 0 \leq t \leq 1.$$

Therefore we can write for each integer $n \geq 1$,

(2.1) $${}^{(k_n)}S_n =_{\mathcal{D}} n \int_0^{U_{n-k_n,n}} K(u) dH_n(u).$$

We now introduce the uniform empirical distribution function

$$G_n(s) = n^{-1} \#\{1 \leq i \leq n : U_i \leq s\}, \ 0 \leq s \leq 1,$$

and the empirical quantile function

$$U_n(s) = \begin{cases} U_{k,n}, & \frac{k-1}{n} < s \leq \frac{k}{n},\ k = 1,\dots,n, \\ U_{1,n}, & s = 0. \end{cases}$$

The following proposition provides a joint weighted approximation of H_n, G_n and U_n, which will be essential to our approach to the asymptotic distribution of $({}^{(k_n)}S_n - B_n)/A_n$.

PROPOSITION. *On a rich enough probability space there exists a sequence of probabilistically equivalent versions $(\tilde{H}_n, \tilde{G}_n, \tilde{U}_n)$ of (H_n, G_n, U_n), meaning that for each integer $n \geq 1$,*

$$(\tilde{H}_n, \tilde{G}_n, \tilde{U}_n) =_{\mathcal{D}} (H_n, G_n, U_n),$$

such that for a Brownian bridge B and a standard Wiener process W, with B and W independent, for all $0 < \nu < \frac{1}{2}$,

$$\sup_{0 \leq t \leq 1} |n^{\frac{1}{2}}\{\tilde{G}_n(t) - t\} - B(t)|/(1-t)^{\frac{1}{2}-\nu} = \mathcal{O}_P(n^{-\nu}), \tag{2.2}$$

$$\sup_{0 \leq t \leq 1-n^{-1}} |n^{\frac{1}{2}}\tilde{H}_n(t) - W(t)|/(1-t)^{\frac{1}{2}-\nu} = \mathcal{O}_P(n^{-\nu}), \tag{2.3}$$

and for all $0 < \delta < \frac{1}{4}$,

$$\sup_{0 \leq t \leq 1-n^{-1}} |n^{\frac{1}{2}}\{t - \tilde{U}_n(t)\} - B(t)|/(1-t)^{\frac{1}{2}-\delta} = \mathcal{O}_P(n^{-\delta}). \tag{2.4}$$

Proof. The proof will be a consequence of the following lemmas. Our first lemma can be derived easily from results and a starting element of the proof in Mason and van Zwet [6].

LEMMA 1. *On a rich enough probability space there exist independent Brownian bridges B_1, B_2 and B_3 and a triangular array of row-wise independent uniform (0,1) random variables $\xi_1^{(n)}(1), \dots, \xi_n^{(n)}(1)$, $\xi_1^{(n)}(2), \dots, \xi_n^{(n)}(2)$, $\xi_1^{(n)}(3), \dots, \xi_n^{(n)}(3)$, $n \geq 1$, such that for all $0 < \nu < \frac{1}{2}$,*

$$\sup_{0 \leq t \leq 1} |n^{\frac{1}{2}}\{G_n^{(i)}(t) - t\} - B_i(t)|/(1-t)^{\frac{1}{2}-\nu} = \mathcal{O}_P(n^{-\nu}) \tag{2.5}$$

and

$$(2.6)\qquad |n^{\frac{1}{2}}\{G_n^{(3)}(\tfrac{1}{2})-\tfrac{1}{2}\}-B_3(\tfrac{1}{2})| = \mathcal{O}_P(n^{-\frac{1}{4}}),$$

where for $i = 1,2,3,\ n \geq 1$,

$$G_n^{(i)}(t) \;=\; n^{-1}\sum_{j=1}^{n} I(\{\xi_j^{(n)}(i) \leq t\}),\ 0 \leq t \leq 1.$$

Set now for $n \geq 1$,

$$N_n = \sum_{j=1}^{n} I(\{\xi_j^{(n)}(3) \leq \tfrac{1}{2}\}) \ \text{ and } \ M_n = n - N_n.$$

Let for $n \geq 1$ and $0 \leq t \leq 1$,

$$n\tilde{G}_n(t) \;=\; N_n G_{N_n}^{(1)}(t) + M_n G_{M_n}^{(2)}(t)$$

and

$$n\tilde{H}_n(t) \;=\; N_n G_{N_n}^{(1)}(t) - M_n G_{M_n}^{(2)}(t).$$

(We define $G_0^{(i)} \equiv 0$ for $i = 1,2$.) Also define

$$B(t) = 2^{-\frac{1}{2}}\{B_1(t) + B_2(t)\},\ 0 \leq t \leq 1,$$

and

$$W(t) = 2^{-\frac{1}{2}}\{B_1(t) - B_2(t)\} + 2tB_3(\tfrac{1}{2}),\ 0 \leq t \leq 1.$$

It is simple to verify that B and W are independent processes with B being a Brownian bridge and W a standard Wiener process.

LEMMA 2. *On the probability space of Lemma 1, for all* $0 < \nu < \frac{1}{2}$,

$$(2.7)\qquad \sup_{0\leq t\leq 1} |n^{\frac{1}{2}}\{\tilde{G}_n(t) - t\} - B(t)|/(1-t)^{\frac{1}{2}-\nu} = \mathcal{O}_P(n^{-\nu})$$

and

$$(2.8)\qquad \sup_{0\leq t\leq 1-n^{-1}} |n^{\frac{1}{2}}\tilde{H}_n(t) - W(t)|/(1-t)^{\frac{1}{2}-\nu} = \mathcal{O}_P(n^{-\nu}).$$

Proof. Obviously, since both

$$n^{-1}N_n \to_P \frac{1}{2} \text{ and } n^{-1}M_n \to_P \frac{1}{2} \text{ as } n \to \infty,$$

we have by (2.5), for all $0 < \nu < \frac{1}{2}$,

$$\sup_{0\le t\le 1} |N_n^{\frac{1}{2}}\{G_{N_n}^{(1)}(t) - t\} - B_1(t)|/(1-t)^{\frac{1}{2}-\nu} = \mathcal{O}_P(n^{-\nu}) \tag{2.9}$$

and

$$\sup_{0\le t\le 1} |M_n^{\frac{1}{2}}\{G_{M_n}^{(2)}(t) - t\} - B_2(t)|/(1-t)^{\frac{1}{2}-\nu} = \mathcal{O}_P(n^{-\nu}). \tag{2.10}$$

Also, by (2.6),

$$\sup_{0\le t\le 1-n^{-1}} \frac{t}{(1-t)^{\frac{1}{2}-\nu}} |\frac{2N_n - n}{n^{\frac{1}{2}}} - 2B_3(\frac{1}{2})| = \mathcal{O}_P(n^{-\nu}). \tag{2.11}$$

Moreover, it is easily checked that

$$2^{-\frac{1}{2}} - (N_n/n)^{\frac{1}{2}} = \mathcal{O}_P(n^{-\frac{1}{2}}) \text{ and } 2^{-\frac{1}{2}} - (M_n/n)^{\frac{1}{2}} = \mathcal{O}_P(n^{-\frac{1}{2}}).$$

In addition, by well-known properties of the Brownian bridge, for $i = 1, 2$,

$$\sup_{0\le t\le 1} |B_i(t)|/(1-t)^{\frac{1}{2}-\nu} = \mathcal{O}_P(1).$$

Thus,

$$\sup_{0\le t\le 1} |(\frac{N_n}{n})^{\frac{1}{2}} B_1(t) - 2^{-\frac{1}{2}} B_1(t)|/(1-t)^{\frac{1}{2}-\nu} = \mathcal{O}_P(n^{-\frac{1}{2}}) \tag{2.12}$$

and

$$\sup_{0\le t\le 1} |(\frac{M_n}{n})^{\frac{1}{2}} B_2(t) - 2^{-\frac{1}{2}} B_2(t)|/(1-t)^{\frac{1}{2}-\nu} = \mathcal{O}_P(n^{-\frac{1}{2}}). \tag{2.13}$$

Using (2.9), (2.10), (2.11), (2.12) and (2.13), a little algebra now gives (2.7) and (2.8). □

For each integer $n \ge 1$, let $\tilde{U}_{1,n} \le \ldots \le \tilde{U}_{n,n}$ be the order statistics of the random variables $\xi_1^{(N_n)}(1), \ldots, \xi_{N_n}^{(N_n)}(1), \xi_1^{(M_n)}(2), \ldots, \xi_{M_n}^{(M_n)}(2)$. Define the empirical quantile function $\tilde{U}_n(t)$ based on these order statistics in the usual way. The following lemma is a consequence of the fact that for all $0 < \delta < \frac{1}{4}$,

$$\sup_{0\le t\le 1-n^{-1}} n^{\delta} |\tilde{G}_n(t) + \tilde{U}_n(t) - 2t|/(1-t)^{\frac{1}{2}-\delta} = \mathcal{O}_P(1),$$

which can be readily inferred from Corollary 2.3 in [2] (also see the Proposition in Mason [5]).

LEMMA 3. *On the probability space of Lemma 1, for all* $0 < \delta < \frac{1}{4}$,

$$\sup_{0 \le t \le 1-n^{-1}} |n^{\frac{1}{2}}\{t - \tilde{U}_n(t)\} - B(t)|/(1-t)^{\frac{1}{2}-\delta} = O_P(n^{-\delta}). \tag{2.14}$$

Since it is routine to verify that for each integer $n \ge 1$, $(\tilde{H}_n, \tilde{G}_n, \tilde{U}_n) =_{\mathcal{D}} (H_n, G_n, U_n)$, assertions (2.2), (2.3) and (2.4) follow from (2.7), (2.8) and (2.14). This completes the proof of the Proposition. □

Returning to the proof of the Theorem, from now on, since we are only concerned with distributional results, on account of (2.1) for the sake of the proof we can and do identify $(\tilde{H}_n, \tilde{G}_n, \tilde{U}_n)$ with (H_n, G_n, U_n); that is, we work on the probability space of the Proposition. Also for notational convenience we drop the prime on the n'. The proof of the Theorem requires a number of lemmas.

LEMMA 4. *Under (1.3), for each* $0 < M < \infty$,

$$\limsup_{n\to\infty} k_n^{\frac{1}{4}} K(u_n(M))/A_n < \infty. \tag{2.15}$$

Proof. Fix any finite $\overline{M} > M$. Then for each $n \ge 1$,

$$A_n^{-2}\tau_n^2(\overline{M}) = A_n^{-2}\tau_n^2(M) + nA_n^{-2}\int_{u_n(M)}^{u_n(\overline{M})} K^2(u)du,$$

from which by letting $n \to \infty$ and using monotonicity of K we get

$$\begin{aligned} v^2(\overline{M}) - v^2(M) &= \lim_{n\to\infty} nA_n^{-2}\int_{u_n(M)}^{u_n(\overline{M})} K^2(u)\,du \\ &\ge \limsup_{n\to\infty} nA_n^{-2}K^2(u_n(M))(\overline{M} - M)k_n^{1/2}n^{-1}, \end{aligned}$$

implying (2.15). □

For each $n \ge 1$, set

$$D_n = \{0 < u_n(-Z_n) < 1\},$$

where

$$Z_n = (\frac{n}{k_n})^{\frac{1}{2}} B(1 - \frac{k_n}{k}) =_{\mathcal{D}} N(0, 1 - \frac{k_n}{n}).$$

Clearly, we have

$$P\{D_n\} \to 1 \text{ as } n \to \infty. \tag{2.16}$$

LEMMA 5. *Under (1.3) we have as $n \to \infty$,*

$$nA_n^{-1} \int_0^{U_{n-k_n,n}} K(u)dH_n(u) = nA_n^{-1} \int_0^{u_n(-Z_n)} K(u)dH_n(u)I(D_n) + o_P(1). \tag{2.17}$$

Proof. Because of (2.16) the assertion will follow from

$$nA_n^{-1} \int_{u_n(-Z_n)}^{U_{n-k_n,n}} K(u)dH_n(u)I(D_n) \to_P 0 \text{ as } n \to \infty. \tag{2.18}$$

Towards a proof of (2.18) notice that from the weighted approximation (2.4) for $U_n(\cdot)$, for any $0 < \delta < \frac{1}{4}$,

$$U_{n-k_n,n} - u_n(-Z_n) = \mathcal{O}_P(k_n^{\frac{1}{2}-\delta}/n), \tag{2.19}$$

which in combination with the fact that $Z_n = \mathcal{O}_P(1)$ gives

$$U_{n-k_n,n} = 1 - \frac{k_n}{n} + \mathcal{O}_P(k_n^{\frac{1}{2}}/n). \tag{2.20}$$

Next, for fixed $0 < \delta < \frac{1}{4}$ and $0 < M < \infty$, set

$$A_n(M) = \{u_n(-M) \le U_{n-k_n,n} \le u_n(M)\}$$

and

$$B_{n,\delta}(M) = \{|U_{n-k_n,n} - u_n(-Z_n)| \le M k_n^{\frac{1}{2}-\delta}/n\}.$$

Then (2.19) and (2.20) imply

$$\lim_{M\to\infty} \liminf_{n\to\infty} P\{A_n(M) \cap B_{n,\delta}(M)\} = 1. \tag{2.21}$$

On the event $D_n \cap A_n(M) \cap B_{n,\delta}(M) =: E_n$, we have for all large n,

$$u_n(-2M) \leq u_n(-Z_n) \leq u_n(2M) \tag{2.22}$$

and we get the estimate after integrating by parts

$$\begin{aligned}|nA_n^{-1} \int_{u_n(-Z_n)}^{U_{n-k_n,n}} K(u) dH_n(u) I(E_n)| &= nA_n^{-1} I(E_n) |K(U_{n-k_n,n})\{H_n(U_{n-k_n,n}) \\ &\quad - H_n(u_n(-Z_n))\} \\ &\quad + \int_{u_n(-Z_n)}^{U_{n-k_n,n}} \{H_n(u_n(-Z_n)) - H_n(u)\} dK(u)| \\ &\leq 2k^{\frac{1}{4}} A_n^{-1} K(u_n(2M)) n k_n^{-\frac{1}{4}} \Delta_n(M),\end{aligned}$$

where

$$\Delta_n(M) = \sup_{u_n(-M) \leq b \leq u_n(M)} \sup_{0 \leq |h| \leq M k_n^{\frac{1}{2}-\delta}/n} |H_n(b+h) - H_n(b)|.$$

In view of Lemma 4, (2.16) and (2.21), the proof of (2.18) and hence of (2.17) will be complete if we show that for all $0 < \delta < \frac{1}{4}$ and $0 < M < \infty$,

$$n k_n^{-\frac{1}{4}} \Delta_n(M) \to_P 0 \quad \text{as} \quad n \to \infty. \tag{2.23}$$

Consider a grid

$$u_n(-M) = b_0 < b_1 < \ldots < b_\ell < \ldots < b_L = u_n(M)$$

such that

$$b_\ell - b_{\ell-1} = M k_n^{\frac{1}{2}-\delta}/n \quad \text{for} \quad \ell = 1, \ldots, L-1$$

and

$$b_L - b_{L-1} \leq M k_n^{\frac{1}{2}-\delta}/n.$$

For each $b \in [u_n(-M), u_n(M))$ there exists exactly one $\ell \in \{1, \ldots, L\}$ such that $b_{\ell-1} \leq b < b_\ell$. This fact gives the bounds

$$\begin{aligned}n k_n^{-\frac{1}{4}} \Delta_n(M) &\leq 2n k_n^{-\frac{1}{4}} \max_{1 \leq \ell \leq L} \sup_{0 \leq |h| \leq 2M k_n^{\frac{1}{2}-\delta}/n} |H_n(b_{\ell-1}+h) - H_n(b_{\ell-1})| \\ &= 2k_n^{-\frac{1}{4}} n^{\frac{1}{2}} \max_{1 \leq \ell \leq L} \omega_n(2M k_n^{\frac{1}{2}-\delta}/n, b_{\ell-1}),\end{aligned}$$

where

$$\omega_n(a,b) = \sup_{0 \le |h| \le a} n^{\frac{1}{2}} |H_n(b+h) - H_n(b)|.$$

To finish the proof of (2.23) it suffices to show that for each fixed $0 < M < \infty$ and $0 < \delta < \frac{1}{4}$ the right side of the last inequality converges to zero in probability. For this, fix $\epsilon > 0$ and note that

$$P\{n^{\frac{1}{2}} k_n^{-\frac{1}{4}} \max_{1 \le \ell \le L} \omega_n(2M k_n^{\frac{1}{2}-\delta}/n, b_{\ell-1}) \ge \epsilon\} \le \sum_{\ell=1}^{L} P\{\omega_n(2M k_n^{\frac{1}{2}-\delta}/n, b_{\ell-1}) \ge \epsilon k_n^{\frac{1}{4}} n^{-\frac{1}{2}}\},$$

which by Theorem 1.1 of Einmahl and Ruymgaart [3] is

$$\le CL \exp\{-\frac{A\epsilon^2}{2M} k_n^{\delta} \psi(\frac{B\epsilon}{4M} k_n^{\delta-\frac{1}{4}})\} \; =: \; I_n(\epsilon, M),$$

where A, B and C are universal positive constants and ψ is an increasing function on $(0,\infty)$ such that $\psi(z) \downarrow 1$ as $z \downarrow 0$. This when combined with the easy bound $L \le 2k_n^{\delta} + 1$ yields $I_n(\epsilon, M) \to 0$ as $n \to \infty$, assuming that $0 < \delta < \frac{1}{4}$. This completes the proof of (2.23) from which (2.17) follows. □

Finally, we require one more lemma.

LEMMA 6. *Under (1.3) we have*

$$\begin{aligned} &nA_n^{-1} \int_0^{u_n(-Z_n)} K(u) dH_n(u) I(D_n) \\ &= A_n^{-1} \left\{ -n^{\frac{1}{2}} \int_0^{u_n(-Z_n)} W(u) dK(u) + n^{\frac{1}{2}} K(u_n(-Z_n)) W(u_n(-Z_n)) \right\} I(D_n) + o_P(1). \end{aligned}$$

Proof. On the event D_n we obtain after integrating by parts that the left side

$$\begin{aligned} &= nA_n^{-1} K(u_n(-Z_n)) H_n(u_n(-Z_n)) - nA_n^{-1} \int_0^{u_n(-Z_n)} H_n(u) dK(u) \\ &= -n^{\frac{1}{2}} A_n^{-1} \int_0^{u_n(-Z_n)} W(u) dK(u) \\ &\quad + n^{\frac{1}{2}} A_n^{-1} \int_0^{u_n(-Z_n)} \{W(u) - n^{\frac{1}{2}} H_n(u)\} dK(u) \\ &\quad + n^{\frac{1}{2}} A_n^{-1} K(u_n(-Z_n)) W(u_n(-Z_n)) \\ &\quad + n^{\frac{1}{2}} A_n^{-1} K(u_n(-Z_n)) \{n^{\frac{1}{2}} H(u_n(-Z_n)) - W(u_n(-Z_n))\}, \end{aligned}$$

so that to finish the proof we must verify that

$$(2.24) \qquad n^{\frac{1}{2}} A_n^{-1} \int_0^{u_n(-Z_n)} \{W(u) - n^{\frac{1}{2}} H_n(u)\} dK(u) I(D_n) \to_P 0 \text{ as } n \to \infty$$

and

$$(2.25) \; n^{\frac{1}{2}} A_n^{-1} K(u_n(-Z_n))\{n^{\frac{1}{2}} H_n(u_n(-Z_n)) - W(u_n(-Z_n))\} I(D_n) \to_P 0 \text{ as } n \to \infty.$$

For the proof of (2.24) fix $0 < \nu < \frac{1}{2}, 0 < \delta < \frac{1}{4}$ and $0 < M < \infty$. In view of (2.22) we obtain on the event E_n

$$\begin{aligned} &|n^{\frac{1}{2}} A_n^{-1} \int_0^{u_n(-Z_n)} \{W(u) - n^{\frac{1}{2}} H_n(u)\} dK(u)| \\ &\leq \left\{ \sup_{0 \leq u \leq 1-n^{-1}} \frac{|W(u) - n^{\frac{1}{2}} H_n(u)|}{(1-u)^{\frac{1}{2}-\nu}} \right\} n^{\frac{1}{2}} A_n^{-1} \int_0^{u_n(2M)} (1-u)^{\frac{1}{2}-\nu} dK(u), \end{aligned}$$

which by (2.3) is equal to

$$(2.26) \qquad \mathcal{O}_P(n^{-\nu}) n^{\frac{1}{2}} A_n^{-1} \int_0^{u_n(2M)} (1-u)^{\frac{1}{2}-\nu} dK(u).$$

Now by integrating by parts and some elementary bounds,

$$\begin{aligned} n^{\frac{1}{2}} A_n^{-1} \int_0^{u_n(2M)} (1-u)^{\frac{1}{2}-\nu} dK(u) \;\leq\; & n^{\frac{1}{2}} A_n^{-1} K(u_n(2M))(k_n/n)^{\frac{1}{2}-\nu} \\ & + (\frac{1}{2} - \nu) n^{\frac{1}{2}} A_n^{-1} \int_0^{u_n(2M)} K(u)(1-u)^{-\nu-\frac{1}{2}} du, \end{aligned}$$

which, by the Cauchy-Schwarz inequality applied to the second term above and by (1.3) and (2.15),

$$\begin{aligned} &\leq n^{\nu} k_n^{\frac{1}{4}-\nu} \{k_n^{\frac{1}{4}} A_n^{-1} K(u_n(2M))\} + \mathcal{O}(1) A_n^{-1} \tau_n(2M)(k_n/n)^{-\nu} \\ &= \mathcal{O}(n^{\nu} k_n^{\frac{1}{4}-\nu}). \end{aligned}$$

This bound when combined with (2.26), (2.16) and (2.21) yields (2.24) as long as $\frac{1}{4} < \nu < \frac{1}{2}$.

To establish (2.25), we fix $\frac{1}{4} < \nu < \frac{1}{2}$ and $0 < M < \infty$. On the event E_n (recall (2.22)),

$$\begin{aligned} &|n^{\frac{1}{2}} A_n^{-1} K(u_n(-Z_n))\{n^{\frac{1}{2}} H_n(u_n(-Z_n)) - W(u_n(-Z_n))\}| \\ &\leq \sup_{0 \leq u \leq 1-n^{-1}} \frac{|n^{\frac{1}{2}} H_n(u) - W(u)|}{(1-u)^{\frac{1}{2}-\nu}} n^{\frac{1}{2}} A_n^{-1} K(u_n(2M))(1 - u_n(-2M))^{\frac{1}{2}-\nu}, \end{aligned}$$

which by (2.3), (2.15) and $\frac{1}{4} < \nu < \frac{1}{2}$,

$$= \mathcal{O}_P(1) k_n^{\frac{1}{4}-\nu} k_n^{\frac{1}{4}} A_n^{-1} K(u_n(2M)) = o_P(1),$$

proving (2.25). This completes the proof of Lemma 6.□

According to Lemmas 5 and 6 combined with (2.1), the assertion of the Theorem will follow if we show that

$$\begin{aligned} V_n &:= A_n^{-1}\{-n^{\frac{1}{2}} \int_0^{u_n(-Z_n)} W(u)dK(u) + n^{\frac{1}{2}} K(u_n(-Z_n))W(u_n(-Z_n)) - B_n\} I(D_n) \\ &\to {}_{\mathcal{D}} V \text{ as } n \to \infty. \end{aligned}$$

We prove this using characteristic functions. Since

$$|E\{\exp(itV_n)(1 - I(D_n))\}| \leq 1 - P(D_n) \to 0 \text{ as } n \to \infty,$$

we have

$$E\{\exp(itV_n)\} = E\{\exp(itV_n)I(D_n)\} + o(1) \text{ as } n \to \infty.$$

Notice that for $x \geq -(1 - k_n/n)nk_n^{-\frac{1}{2}}$ the random variable

$$-n^{\frac{1}{2}} \int_0^{u_n(x)} W(u)dK(u) + n^{\frac{1}{2}} K(u_n(x))W(u_n(x))$$

is normally distributed with mean zero and variance $\tau_n^2(x)$. Therefore by the independence of Z_n and W, for any t,

$$E\{\exp(itV_n)I(D_n)\} = \exp(-itA_n^{-1}B_n) \int_{-(1-\frac{k_n}{n})\frac{n}{\sqrt{k_n}}}^{\infty} \exp(-\frac{v_n^2(x)t^2}{2}) dP(-Z_n \leq x).$$

From

$$-Z_n \to_{\mathcal{D}} N(0,1) \text{ and } -(1 - \frac{k_n}{n})\frac{n}{\sqrt{k_n}} \to -\infty \text{ as } n \to \infty$$

it follows by conditions (1.2) and (1.3) that for any t,

$$E\{\exp(itV_n)I(D_n)\} \to \exp(iCt) \int_{-\infty}^{\infty} \exp(-\frac{v^2(x)}{2}t) d\Phi(x),$$

completing the proof of the Theorem. □

REFERENCES

[1] CSÖRGŐ, S., HAEUSLER, E. and MASON, D.M. (1988). A probabilistic approach to the asymptotic distribution of sums of independent, identically distributed random variables. *Adv. in Appl. Math.* **9**, 259-333.

[2] CSÖRGŐ, M., CSÖRGŐ, S., HORVÁTH, L. and MASON, D.M. (1986). Weighted empirical and quantile processes. *Ann. Probab.* **14**, 31-85.

[3] EINMAHL, J.H.J. and RUYMGAART, F.H. (1986). Some properties of weighted compound multivariate empirical processes. *Sankhyā, Series A* **48**, 393-403.

[4] GRIFFIN, P.S. and PRUITT, W.E. (1987). The central limit theorem for trimmed sums. *Math. Proc. Camb. Phil. Soc.* **102**, 329-349.

[5] MASON, D.M. (1990). A note on weighted empirical and quantile processes. This volume.

[6] MASON, D.M. and VAN ZWET, W.R. (1987). A refinement of the KMT inequality for the uniform empirical process. *Ann. Probab.* **15**, 871-884.

[7] PRUITT, W.E. (1988). Sums of independent random variables with the extreme terms excluded. In: *Probability and Statistics. Essays in Honor of Franklin A. Graybill* (J.N. Srivastava, Ed.), pp. 201-216, Elsevier, Amsterdam.