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INTERMEDIATE SUMS AND STOCHASTIC COMPACTNESS OF MAXIMA

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Let $X_{1,n} \leq \dots \leq X_{n,n}$ be the order statistics of n independent random variables with a common distribution function F and let k_n be positive numbers such that $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$, and consider the sums $I_n(a,b) = \sum_{i=\lceil ak_n\rceil+1}^{\lceil bk_n\rceil} X_{n+1-i,n}$ of intermediate order statistics, where $0 < a < b$. We find necessary and sufficient conditions for the existence of constants $A_n > 0$ and C_n such that $A_n^{-1}(I_n(a,b)-C_n)$ converges in distribution along subsequences of the positive integers {n} to non-degenerate limits and completely describe the possible subsequential limiting distributions. We also disclose an unexpected connection between the convergence in distribution of the intermediate sums $I_n(a,b)$ and the stochastic compactness of the maxima $X_{n,n}$.

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1. Introduction and Statement of Results. Let $X, X_1, X_2, \ldots$ be a sequence of independent non-degenerate random variables with a common distribution function $F(x) = P\{X \leq x\}$, $x \in \mathbb{R}$, and for each integer $n \geq 1$ let $X_{1,n} \leq \ldots \leq X_{n,n}$ denote the order statistics based on the sample $X_1, \ldots, X_n$. Let $\{k_n\}$ be a sequence of positive numbers such that

$$(1.1) \qquad k_n \to \infty \quad \text{and} \quad k_n/n \to 0 \quad \text{as } n \to \infty.$$

When the k_n are integers, many authors have investigated the asymptotic distribution of the single intermediate order statistic $X_{n+1-k_n,n}$. (See Pickands (1975), Balkema and de Haan (1978), Watts, Rootzén and Leadbetter (1982) and Cooil (1985) for example, and the references therein.) In this paper we are interested in the problem of the asymptotic distribution of the intermediate sum

$$(1.2) \qquad I_n(a,b) = I_n(a,b;k_n) = \sum_{i=\lceil ak_n \rceil+1}^{\lceil bk_n \rceil} X_{n+1-i,n}, \quad 0 < a < b,$$

where $\lceil x \rceil$ is the smallest integer not smaller than x, $\{k_n\}$ satisfies (1.1) and the empty sum is always understood as zero.

Consider the inverse or quantile function of F defined as

$$(1.3) \qquad Q(s) = \inf\{x : F(x) \geq s\}, \quad 0 < s \leq 1.$$

It will be more convenient in our investigations to work with the left continuous non-decreasing function

$$(1.4) \qquad H(s) = -Q((1-s)-), \quad 0 \leq s < 1,$$

the increments

$$(1.5) \qquad \Delta_n(a,b) = H\left(\frac{\lceil bk_n \rceil}{n}\right) - H\left(\frac{\lceil ak_n \rceil}{n}\right), \quad 0 < a < b,$$

of which will play an important role. Set

$$(1.6) \qquad \Delta_n^*(a,b) = \begin{cases} \Delta_n(a,b), & \text{if } \Delta_n(a,b) > 0, \\ 1, & \text{if } \Delta_n(a,b) = 0, \end{cases}$$

and for c = a,b and n large enough define the functions

$$\psi_n(c;x) = \begin{cases} \dfrac{H\left(\frac{\lceil ck_n \rceil}{n} + x\frac{\sqrt{k_n}}{n}\right) - H\left(\frac{\lceil ck_n \rceil}{n}\right)}{\Delta_n^*(a,b)}, & -\frac{c\sqrt{k_n}}{2} \le x \le \frac{c\sqrt{k_n}}{2}, \\ \psi_n(-c\sqrt{k_n}/2), & -\infty < x < -c\sqrt{k_n}/2, \\ \psi_n(c\sqrt{k_n}/2), & c\sqrt{k_n}/2 < x < \infty. \end{cases}$$

These are non-decreasing and left-continuous for each n, and satisfy $\psi_n(c;0) = 0$. We also need the functions

$$\varphi_n(x) = \begin{cases} 0, & x < \frac{\lceil ak_n \rceil}{k_n} \\ \left\{H\left(\frac{xk_n}{n}\right) - H\left(\frac{\lceil ak_n \rceil}{n}\right)\right\} / \Delta_n^*(a,b), & \frac{\lceil ak_n \rceil}{k_n} \le x \le \frac{\lceil bk_n \rceil}{k_n}, \\ \varphi_n\left(\frac{\lceil bk_n \rceil}{k_n}\right), & x > \frac{\lceil bk_n \rceil}{k_n}. \end{cases}$$

Note that $\varphi_n \equiv 0$ on $\mathbb{R}$ whenever $\Delta_n(a,b) = 0$, otherwise φ_n is a left-continuous distribution function on $\mathbb{R}$ for which $\varphi_n(x) = 1$ if $x \ge \lceil bk_n \rceil / k_n$. In what follows, the symbol => will denote weak convergence of functions, that is, pointwise convergence at every continuity point of the limiting function in the interval to be indicated, and $\to_{\mathcal{D}}$ will denote convergence in distribution. Finally, introduce the centering sequence

$$(1.7) \qquad \mu_n(a,b) = -n \int_{\lceil ak_n \rceil/n}^{\lceil bk_n \rceil/n} H(s)\, ds$$

and, as usual, let W(t), t ≥ 0, denote a standard Wiener process.

THEOREM 1. *Let* $\{k_n\}$ *be a sequence as in* (1.1) *and fix* $0 < a < b$.

(i) *Suppose that there exist a subsequence* $\{n'\}$ *of the positive intergers* $\{n\}$ *and non-decreasing, left-continuous functions* ψ_a *and* ψ_b *on* $\mathbb{R}$ *such that* $\psi_c(0) \le 0$, $\psi_c(0+) \ge 0$, *and*

$$(1.8) \qquad \psi_{n'}(c;\cdot) \Rightarrow \psi_c(\cdot) \text{ on } \mathbb{R} \text{ as } n' \to \infty, \quad c = a,b.$$

Then there exist a subsequence $\{n''\} \subset \{n'\}$ *and a non-negative, non-decreasing, left-continuous function* φ *on* $\mathbb{R}$ *such that*

$$(1.9) \quad \varphi(a) = 0, \quad \text{either } \varphi(\infty) = 0 \text{ or } \varphi(b+) = 1, \quad \text{and} \quad \varphi_{n''}(\cdot) \Rightarrow \varphi(\cdot) \text{ on } \mathbb{R}$$

and

$$(1.10) \quad \frac{1}{\sqrt{k_{n''}}\,\Delta^*_{n''}(a,b)} \left\{ \sum_{i=\lceil ak_{n''}\rceil+1}^{\lceil bk_{n''}\rceil} X_{n''+1-i,n''} - \mu_{n''}(a,b) \right\} \to_{\mathcal{D}} V(\psi_a,\varphi,\psi_b)$$

as $n'' \to \infty$, *where*

$$V(\psi_a,\varphi,\psi_b) = \int_0^{-W(a)} \psi_a(x)dx + \int_{[a,b]} W(x)d\varphi(x) - \int_0^{-W(b)} \psi_b(x)dx$$

$$= -\int_{-W(a)}^{0} \psi_a(x)dx + \int_a^b W(x)d\varphi(x) + (\varphi(b+) - \varphi(b))W(b) + \int_{-W(b)}^{0} \psi_b(x)dx.$$

Furthermore, we necessarily have

$$(1.11) \qquad \psi_a(x) \le \varphi(a+) \quad \text{and} \quad \psi_b(x) \ge \varphi(b)-1, \quad x \in \mathbb{R},$$

and the limiting random variable $V(\psi_a,\varphi,\psi_b)$ *is degenerate if and only if* $\psi_a = \psi_b \equiv 0$ *on* $\mathbb{R}$ *and* $\varphi \equiv 0$ *on* $[a,b]$.

(ii) *If for some subsequence* $\{n'\} \subset \{n\}$ *and constants* $B_{n'} > 0$ *there exist non-decreasing, left-continuous functions* ψ_a *and* ψ_b *on* $\mathbb{R}$ *such that* $\psi_c(0) \le 0$, $\psi_c(0+) \ge 0$, *and*

$$(1.12) \qquad \frac{\Delta^*_{n'}(a,b)}{B_{n'}} \psi_{n'}(c;\cdot) \Rightarrow \psi_c(\cdot) \text{ on } \mathbb{R}, \ c=a,b,$$

and

(1.13) $$\Delta_{n'}(a,b)/B_{n'} \to 0 \quad as \quad n' \to \infty$$

then

(1.14) $$\frac{1}{\sqrt{k_{n'}}\, B_{n'}} \left\{ \sum_{i=\lceil ak_{n'}\rceil+1}^{\lceil bk_{n'}\rceil} X_{n'+1-i,n'} - \mu_{n'}(a,b) \right\} \to_{\mathscr{D}} V(\psi_a, 0, \psi_b)$$

as $n' \to \infty$. *Furthermore, we necessarily have*

(1.15) $$\psi_a(x) = 0, \quad x \geq 0, \quad and \quad \psi_b(x) = 0, \quad x < 0,$$

and the limiting random variable $V(\psi_a, 0, \psi_b)$ *is degenerate if and only if* $\psi_a = \psi_b \equiv 0$.

The following converse result shows that Theorem 1 is optimal in general.

THEOREM 2. *Suppose that there exist a subsequence* $\{n'\} \subset \{n\}$ *and norming and centering constants* $A_{n'} > 0$ *and* $C_{n'}$ *such that*

(1.16) $$\frac{1}{A_{n'}} \left\{ \sum_{i=\lceil ak_{n'}\rceil+1}^{\lceil bk_{n'}\rceil} X_{n'+1-i,n'} - C_{n'} \right\} \to_{\mathscr{D}} V^{*} \quad as \quad n' \to \infty$$

where V^{*} *is a non-degenerate random variable. Then there exist a subsequence* $\{n''\} \subset \{n'\}$ *and non-decreasing, left-continuous functions* ψ_a *and* ψ_b *on* $\mathbb{R}$ *such that* $\psi_c(0) \leq 0$, $\psi_c(0+) \geq 0$, *and*

(1.17) $$\psi^{*}_{n''}(c;\cdot) = \frac{\sqrt{k_{n''}}\, \Delta^{*}_{n''}(a,b)}{A_{n''}} \psi_{n''}(c;\cdot) \Rightarrow \psi_c(\cdot) \text{ on } \mathbb{R}, \quad c=a,b,$$

and for some $0 \leq \delta < \infty$,

(1.18) $$\sqrt{k_{n''}}\, \Delta_{n''}(a,b)/A_{n''} \to \delta \quad as \quad n'' \to \infty.$$

For the limiting random variable V^{*} *we necessarily have the distributional equality*

(1.19) $$V^* =_{\mathfrak{D}} V(\psi_a, \delta\varphi, \psi_b) + \gamma,$$

where $\gamma \in \mathbb{R}$ *is some constant.*

Our last theorem establishes an unexpected connection between the convergence in distribution of intermediate sums $I_n(a,b)$ and stochastic compactness of the maximum observation $X_{n,n}$.

We say that $\{X_{n,n}\}$ is stochastically compact of there exist a sequence of norming constants $a(n) > 0$ and a sequence of centering constants $c(n)$ such that for every subsequence $\{n'\} \subset \{n\}$ there exists a further subsequence $\{n''\} \subset \{n'\}$ such that $(X_{n'',n''} - c(n''))/a(n'')$ converges in distribution to a non-degenerate limit. Let $\lfloor x \rfloor$ be the largest integer not greater than x, the usual integer part of $x \in \mathbb{R}$, and for a number $0 < s < 1$ set

$$h_s(x) = \begin{cases} H(sx) - H(s) \quad , & 0 < x < 1/(2s), \\ H(1/2) - H(s) \; , & 1/(2s) \le x < \infty \; . \end{cases}$$

Then it can be easily inferred from the Proposition of de Haan and Resnick (1984), page 590, that $\{X_{n,n}\}$ is stochastically compact if and only if there exist a sequence of constants $B(n) > 0$ such that for every sequence k_n of positive numbers satisfying (1.1) and for every subsequence $\{n'\} \subset \{n\}$ there exist a further subsequence $\{n''\} \subset \{n'\}$ and a non-decreasing, left-continuous finite function $h(x)$, $x > 0$, such that $h(1) \le 0$, $h(1+) \ge 0$, $h \not\equiv 0$, and

(1.20) $$h_{k_{n''}/n''}(\cdot)/B(\lfloor n''/k_{n''} \rfloor) \Rightarrow h(\cdot) \quad \text{on} \quad (0,\infty)$$

as $n'' \to \infty$, in which case $\{X_{n,n}\}$ is stochastically compact with norming constants $B(n)$ and centering constants $c(n) = -H(1/n)$. The "asymptotic balance" condition (1.20) and its equivalent forms are investigated in depth by de Haan and Resnick (1984) who also provide many interesting examples.

THEOREM 3. *The following two statements are equivalent:*

(I) *There exists a sequence of norming constants* $B(n) > 0$ *such that for* *every sequence* $\{k_n\}$ *of positive numbers satisfying (1.1) and subsequence* $\{n'\} \subset \{n\}$ *there exist a further subsequence* $\{n''\} \subset \{n'\}$ *and numbers* $0 < a_0 < b_0 < \infty$ *such that for each* $0 < a < a_0 < b_0 < b < \infty$ *and subsequence* $\{n'''\} \subset \{n''\}$ *there is a further subsequence* $\{n^{iv}\} \subset \{n'''\}$ *for which*

$$(1.21)\quad \frac{1}{\sqrt{k_{n^{iv}}}\, B\left[\left\lfloor n^{iv}/k_{n^{iv}}\right\rfloor\right]} \left\{ \sum_{i=\lceil ak_{n^{iv}}\rceil+1}^{\lceil bk_{n^{iv}}\rceil} X_{n^{iv}+1-i,n^{iv}} - \mu_{n^{iv}}(a,b) \right\} \to_{\mathcal{D}} Y$$

as $n^{iv} \to \infty$, *where* Y *is a non-degenerate random variable and* $\mu_n(a,b)$ *is as in (1.7).*

(II) $\{X_{n,n}\}$ *is stochastically compact with norming constants* $B(n)$ *and centering constants* $-H(1/n)$.

We close by an example. We say that F is in the domain of attraction of an extreme value distribution if $(X_{n,n}-c_n)/a_n \to_{\mathcal{D}} Y$ as $n \to \infty$, where $a_n > 0$ and $c_n \in \mathbb{R}$ are some constants and Y is non-degenerate. As pointed out in Csörgő, Haeusler, and Mason (1990), with earlier references, this happens if and only if for some constant $\gamma \in \mathbb{R}$,

$$(1.22)\qquad \lim_{s\downarrow 0} \frac{H(sx)-H(sy)}{H(sv)-H(sw)} = \begin{cases} (x^{-\gamma}-y^{-\gamma})/(v^{-\gamma}-w^{-\gamma}) & \text{if } \gamma \neq 0, \\ \log(x/y)/\log(v/w)\ , & \text{if } \gamma = 0, \end{cases}$$

for all distinct $0 < x,y,v,w < \infty$. In this case we write $F \in \Lambda_\gamma$, where, with appropriate choices of a_n and c_n,

$$\Lambda_\gamma(y) = P\{Y \le y\} = \begin{cases} \exp(-y^{1/\gamma}) \quad , & y > 0; \quad \text{if } \gamma > 0, \\ \exp(-\exp(-y)), & y \in \mathbb{R}; \quad \text{if } \gamma = 0, \\ \exp(-(-y)^{1/\gamma}), & y < 0; \quad \text{if } \gamma < 0. \end{cases}$$

It is easily checked that if $F \in \Lambda_\gamma$ for some $\gamma \in \mathbb{R}$, then for all $x \in \mathbb{R}$, $\psi_n(c;x) \to 0$ for any choice of $c > 0$ and $\varphi_n(x) \to \varphi_\gamma(x)$, $a \le x \le b$, for any choice of $0 < a < b$ as $n \to \infty$, where

$$\varphi_\gamma(x) = \varphi_{\gamma,a,b}(x) = \begin{cases} (x^{-\gamma}-a^{-\gamma})/(b^{-\gamma}-a^{-\gamma}), & \text{if } \gamma \neq 0, \\ \log(x/a)/\log(b/a), & \text{if } \gamma = 0. \end{cases}$$

The last convergence follows, of course, from (1.22). So if $F \in \Lambda_\gamma$ for some $\gamma \in \mathbb{R}$, then by Theorem 1,

$$\frac{1}{\sqrt{k_n}\,\Delta_n^*(a,b)} \left\{ \sum_{i=\lceil ak_n\rceil+1}^{\lceil bk_n\rceil} X_{n+1-i,n} - \mu_n(a,b) \right\} \to_{\mathcal{D}} \int_a^b W(x)\, d\varphi_\gamma(x)$$

as $n \to \infty$ for any choice of $0 < a < b < \infty$ and sequence $\{k_n\}$ as in (1.1).

2. **Proofs.** Let $U_1,\dots,U_n$ be independent random variables uniformly distributed on (0,1) with corresponding order statistics $U_{1,n} \le \dots \le U_{n,n}$. Consider the uniform empirical and quantile processes $\alpha_n(t) = \sqrt{n}(\tilde{G}_n(t)-t)$ and $\beta_n(t) = \sqrt{n}(t-\tilde{U}_n(t))$, $0 \le t \le 1$, where $\tilde{G}_n(t) = n^{-1}\,\#\{1 \le k \le n:\ U_k \le t\}$, $0 \le t \le 1$, and $\tilde{U}_n(t) = \inf\{0 \le s \le 1 : \tilde{G}_n(s) \ge t\}$, $0 < t \le 1$, $\tilde{U}_n(0) = U_{1,n}$, so that $\tilde{U}_n(t) = U_{k,n}$ if $(k-1)/n < t \le k/n$, $k=1,\dots,n$. The tail empirical process (cf. Mason (1988)) and the tail quantile process (cf. Cooil (1985)) pertaining to the given sequence $\{k_n\}$ satisfying (1.1) are defined as $\tilde{w}_n(s) = (n/k_n)^{1/2}\,\alpha_n(sk_n/n)$ and $\tilde{v}_n(s) = (n/k_n)^{1/2}\,\beta_n(sk_n/n)$, $0 \le s \le n/k_n$. As Mason (1988) points out, for any $T > 0$, $\tilde{w}_n(\cdot)$ converges weakly in the Skorohod space $D[0,T]$ to a standard Wiener process. Then by a Skorohod construction there exist versions $w_n(\cdot)$ of $\tilde{w}_n(\cdot)$ on a suitable probability space such that for any $T > 0$ we have the distributional equalities

(2.1) $$\{w_n(s) : 0 \le s \le T\} =_{\mathcal{D}} \{\tilde{w}_n(s) : 0 \le s \le T\}$$

for each n large enough to have $n/k_n \ge T$ and

(2.2) $$\sup_{0\le s\le T} |w_n(s) - W(s)| \to 0 \quad \text{a.s.} \quad \text{as } n \to \infty,$$

where $W(s)$, $s \ge 0$ is a standard Wiener process. Then by the obvious left-continuous version of Lemma 1 of Vervaat (1972) we obtain the versions $v_n(\cdot)$ of $\tilde{v}_n(\cdot)$ defined on the same suitable space such that

(2.3) $$\{v_n(s) : 0 \leq s \leq T\} =_{\mathcal{D}} \{\tilde{v}_n(s) : 0 \leq s \leq T\}$$

for all n such that $n/k_n \geq T$ and

(2.4) $$\sup_{0 \leq s \leq T} |v_n(s) - W(s)| \to 0 \quad \text{a.s.} \quad \text{as } n \to \infty.$$

From the definition of H in (1.4) and (1.3) we have

$$(X_{1,n}, \ldots, X_{n,n}) =_{\mathcal{D}} (-H(U_{n,n}), \ldots, -H(U_{1,n})), \; n \geq 1.$$

If we integrate with respect to a right-continuous or a left-continuous function, we accordingly use the symbol $\int_x^y$ to mean $\int_{(x,y]}$ or $\int_{[x,y)}$. Using this convention (in fact already used in (1.7) and in the formulation of the theorems), the notations in (1.2) and (1.7), integration by parts and some elementary rearrangements, from the last distributional equality we obtain

$$I_n(a,b) - \mu_n(a,b) =_{\mathcal{D}} n \int_{\lceil ak_n \rceil/n}^{\lceil bk_n \rceil/n} (\tilde{G}_n(u)-u)dH(u)$$

$$- n \int_{\lceil ak_n \rceil/n}^{\tilde{U}_n(ak_n/n)} (\tilde{G}_n(u)-u)dH(u)$$

$$+ n \int_{\lceil bk_n \rceil/n}^{\tilde{U}_n(bk_n/n)} (\tilde{G}_n(u)-u)dH(u).$$

Substituting $u=sk_n/n$ and going over to the probability space of relations (2.1)-(2.4), we arrive at

(2.5) $$I_n(a,b) - \mu_n(a,b) =_{\mathcal{D}} M_n - R_n(a) + R_n(b),$$

where

$$M_n = \sqrt{k_n} \int_{\lceil ak_n \rceil/k_n}^{\lceil bk_n \rceil/k_n} w_n(s)dH\left(\frac{sk_n}{n}\right)$$

and

$$R_n(c) = \int_{\frac{\lceil ck_n \rceil}{k_n}}^{-\frac{1}{\sqrt{k_n}} v_n\left(\frac{\lceil ck_n \rceil}{k_n}\right) + \frac{\lceil ck_n \rceil}{k_n}} \{\sqrt{k_n}\, w_n(s) + sk_n - \lceil ck_n \rceil\}\, dH\left(\frac{sk_n}{n}\right), \quad c = a,b.$$

Aiming at a proof of Theorem 1, we handle these terms in separate lemmas.

LEMMA 1. *For any subsequence* $\{n'\} \subset \{n\}$ *there exist a further subsequence* $\{n''\} \subset \{n'\}$ *and a non-negative, non-decreasing, left-continuous function* φ *such that* (1.9) *holds and*

$$\frac{M_{n''}}{\sqrt{k_{n''}}\ \Delta^{*}_{n''}(a,b)} \to \int_{[a,b]} W(s)d\varphi(s) = \int_a^b W(s)d\varphi(s) + (\varphi(b+)-\varphi(b))W(b)$$

almost surely as $n'' \to \infty$.

Proof. We distinguish two cases. In the first one there exists a subsequence $\{n''\} \subset \{n'\}$ such that $\Delta_{n''}(a,b) = 0$ for all n''. In this case $\varphi_{n''} \equiv 0$ on $\mathbb{R}$, and hence we have (1.9) with $\varphi \equiv 0$ on $\mathbb{R}$, implying the second statement with zero limit.

The second case is when $\Delta_{n'}(a,b) > 0$ for all n' large enough. In this case, since $\varphi_{n'}(a) = \varphi_{n'}(\lceil ak_{n'} \rceil / k_{n'}) = 0$, $\varphi_{n'}(\lceil bk_{n'} \rceil / k_{n'}) = 1$, $\lceil bk_{n'} \rceil / k_{n'} \geq b$, $\lceil bk_{n'} \rceil / k_{n'} \to b$ as $n' \to \infty$, by a Helly selection we can choose a subsequence $\{n''\} \subset \{n'\}$ such that (1.9) holds with $\varphi(b+) = 1$.

To prove the second statement in the present second case, notice that

$$\frac{M_{n''}}{\sqrt{k_{n''}}\ \Delta^{*}_{n''}(a,b)} = \int_0^\infty w_{n''}(s)d\varphi_{n''}(s)$$

and

$$\left| \int_0^\infty w_{n''}(s)d\varphi_{n''}(s) - \int_0^\infty W(s)d\varphi_{n''}(s) \right| \leq \sup_{0 \leq s \leq 2b} |w_{n''}(s) - W(s)|$$

for all large enough n'', and this bound goes to zero almost surely as $n'' \to \infty$ by (2.2). Furthermore, if $a < d < b$ is a continuity point of φ, then, since $\varphi_{n''}(a) = 0$ and $\varphi(a) = 0$ and since for

$$T_n(b) = \int_b^{\lceil bk_n \rceil / k_n} (W(s) - W(b)) d\varphi_n(s)$$

we have

$$|T_n(b)| \le \sup_{b \le s \le \lceil bk_n \rceil / k_n} |W(s) - W(b)| \to 0$$

almost surely as $n \to \infty$, we obtain that

$$\int_0^\infty W(s) d\varphi_{n''}(s) = \int_a^d (W(s) - W(a))\, d\varphi_{n''}(s) + W(a)\, \varphi_{n''}(d)$$

$$+ \int_d^b (W(s) - W(b))\, d\varphi_{n''}(s) + W(b)(1 - \varphi_{n''}(d))$$

$$+ T_{n''}(b)$$

converges almost surely to

$$\int_a^d (W(s) - W(a)) d\varphi(s) + W(a)\varphi(d) + \int_d^b (W(s) - W(b))\, d\varphi(s) + W(b)(1 - \varphi(d))$$

$$= \int_a^b W(s) d\varphi(s) + W(b)(1 - \varphi(b)) = \int_{[a,b]} W(s) d\varphi(s). \qquad \square$$

LEMMA 2. *If (1.12) holds for some sequence* $B_{n'} > 0$ *(that can be* $\Delta^*_{n'}(a,b)$*) along* $\{n'\}$ *with limiting functions* ψ_a *and* ψ_b *having the properties listed above (1.12), then for* $c = a, b$,

$$\frac{R_{n'}(c)}{\sqrt{k_{n'}}\, B_{n'}} \to \int_{-W(c)}^0 \psi_c(x)\, dx$$

almost surely as $n' \to \infty$.

Proof. By the change of variables

$$s = \frac{\lceil ck_n \rceil}{k_n} + \frac{x}{\sqrt{k_n}},$$

where c is either a or b, we obtain

$$\frac{R_{n'}(c)}{\sqrt{k_{n'}}\, B_{n'}}$$

$$= \int_0^{-v_{n'}(\lceil ck_{n'}\rceil / k_{n'})} \left\{ w_{n'}\left(\frac{\lceil ck_{n'}\rceil}{k_{n'}} + \frac{x}{\sqrt{k_{n'}}}\right) + x \right\} d\, \frac{H\left(\frac{\lceil ck_{n'}\rceil}{n'} + x\, \frac{\sqrt{k_{n'}}}{n'}\right) - H\left(\frac{\lceil ck_{n'}\rceil}{n'}\right)}{B_{n'}}.$$

Therefore, using (2.2) and (2.4) for T=2c, say, and the almost sure uniform continuity of $W(\cdot)$ on $[0,2c]$, it follows that

$$\begin{aligned}\frac{R_{n'}(c)}{\sqrt{k_{n'}}\, B_{n'}} &= \int_0^{-W(c)+o(1)} \{W(c) + o(1) + x\}\, d\psi^*_{n'}(c;x) \\ &= \int_0^{-W(c)} \{W(c) + x\}\, d\psi^*_{n'}(c;x) + o(1) \\ &= \int_0^{-W(c)} x\, d\psi^*_{n'}(c;x) + W(c)\, \psi^*_{n'}(c;-W(c)) + o(1) \\ &= \int_0^{-W(c)} x\, d\psi_c(x) + W(c)\, \psi_c(-W(c)) + o(1) \\ &= \int_{-W(c)}^{0} \psi_c(x)\, dx + o(1)\end{aligned}$$

almost surely as $n' \to \infty$, where we also used the fact that $-W(c)$ is almost surely a continuity point of $\psi_c(\cdot)$. □

The next lemma is needed for the proof of the degeneracy statements in Theorem 1.

LEMMA 3. *Let* g *and* h *be two Borel measurable functions on* $\mathbb{R}$ *and* Z_1 *and*

Z_2 *be two independent non-degenerate normal random variables. If* $g(Z_1) = h(Z_1+Z_2)$ *almost surely, then* g *and* h *are constant almost everywhere on* $\mathbb{R}$.

Proof. The condition implies that $P\{g(Z_1) = h(Z_1+Z_2)|Z_1=z\} = 1$ for almost all $z \in \mathbb{R}$, which in turn implies that $P\{g(z) = h(z + Z_2)\} = 1$ for almost all $z \in \mathbb{R}$. Thus $g(z) = h(x)$ for almost every z and x, which occurs only if both g and h are the same constant almost everywhere. □

Proof of Theorem 1. All the statements in (1.9), (1.10) and in (1.14) follow directly from the distributional equality (2.5) and Lemmas 1 and 2.

To prove (1.11) in case (i), let $x \in \mathbb{R}$ be arbitrary and consider a continuity point s of φ in (a,b). Then, since H is non-decreasing, $\psi_{n''}(a;x) \leq \varphi_{n''}(s)$ for all n" large enough. Hence, letting $n'' \to \infty$, $\limsup \psi_{n''}(a;x) \leq \varphi(s)$, which letting now $s \downarrow a$, implies the statement for ψ_a. Similarly, for all $x \in \mathbb{R}$ and n" large enough,

$$\psi_{n''}(b;x) \geq \{H(sk_{n''}/n'') - H(\lceil bk_{n''}\rceil/n'')\}/\Delta^*_{n''}(a,b)$$
$$= \varphi_{n''}(s) - 1,$$

and the statement for ψ_b follows by letting first $n'' \to \infty$ and then $s \uparrow b$.

The corresponding statement (1.15) in case (ii) follows in exactly the same way, using also (1.13).

Finally we prove the claims about the degeneracy of the limiting random variables. If all ψ_a, φ, and ψ_b are identically zero, then $V(\psi_a,\varphi,\psi_b) = 0$ almost surely. To prove the converse statements, set

$$g_c(y) = \int_0^{-y} \psi_c(x)dx, \quad y \in \mathbb{R}, \quad c = a,b.$$

Assume that in case (ii), $V(\psi_a,0,\psi_b) = C$ almost surely for some $C \in \mathbb{R}$. Writing $Z_1 = W(a)$, $Z_2 = W(b)-W(a)$, $g = g_a$, and $h=g_b+C$, the degeneracy statement follows directly from Lemma 3.

Assume now that $V(\psi_a, \varphi, \psi_b) = C$ almost surely in case (i). Using the notations just introduced, this means that

(2.6) $$g_a(Z_1) + \gamma_1 Z_1 + Z_3 + g_b(Z_1+Z_2) - C + \gamma_2 Z_2 = 0$$

almost surely, where $\gamma_1 = \varphi(b+) - \varphi(a)$,

$$\gamma_2 = \int_{[a,b]} (u-a)\, d\varphi(u)/(b-a),$$

and

$$Z_3 = \int_{[a,b]} W(u)\, d\varphi(u) - \gamma_1\, W(a) - \gamma_2 (W(b) - W(a))$$

$$= \int_{[a,b]} W(u)\, d\varphi(u) - \gamma_1\, Z_1 - \gamma_2\, Z_2,$$

so that Z_1, Z_2, and Z_3 are independent. Therefore, (2.6) can happen only if both Z_3 and $g_a(Z_1) + \gamma_1 Z_1 + g_b(Z_1+Z_2) - C + \gamma_2 Z_2$ are degenerate. However, since $\varphi(a) = 0$, Z_3 can degenerate only if $\varphi \equiv 0$ on $[a,b]$. But in this case $\gamma_1 = \gamma_2 = 0$, and hence Lemma 3 implies again that $\psi_a = \psi_b \equiv 0$ on $\mathbb{R}$ exactly as in case (ii). The theorem is completely proved. □

In the proof of Theorem 2 we will use the representation in (2.5) in the form

(2.7) $$I_n(a,b) - \mu_n(a,b) =_{\mathcal{D}} M_n + R_n(b) - R_n(a),$$

where for c = a,b, as seen in the proof of Lemma 2,

$$R_n(c) = \int_0^{-v_n(\lceil ck_n\rceil/k_n)} \left\{ w_n\left(\frac{\lceil ck_n\rceil}{k_n} + \frac{x}{\sqrt{k_n}} \right) + x \right\} dH\left(\frac{\lceil ck_n\rceil}{n} + x\, \frac{\sqrt{k_n}}{n} \right)$$

and, transforming back the tail empirical process,

$$M_n = \int_{\lceil ak_n\rceil/n}^{\lceil bk_n\rceil/n} n(G_n(u)-u)\, dH(u),$$

where

$$G_n(u) = u + \sqrt{k_n}\, w_n(nu/k_n)/n, \quad 0 \le u \le 1,$$

for which we have $\{G_n(u) : 0 \le u \le 1\} =_{\mathcal{D}} \{\tilde{G}_n(u) : 0 \le u \le 1\}$.

We begin by establishing several lemmas needed in the proof.

LEMMA 4. *For* $D_n(a,b) = |R_n(b) - R_n(a)|/(\sqrt{k_n}\, \Delta_n^*(a,b))$ *we have*

$$\lim_{M\to\infty} \liminf_{n\to\infty} P\{D_n(a,b) < M\} > 0.$$

Proof. In the course of the proof of (1.11) we have already shown that

(2.8) $$\limsup_{n\to\infty} \psi_n(a;x) \le 1 \quad \text{for } x > 0$$

and

(2.9) $$\limsup_{n\to\infty}(-\psi_n(b;x)) \le 1 \quad \text{for } x < 0.$$

Also, as in the proof of Lemma 2,

(2.10) $$D_n(a,b) = \left| \int_0^{-W(b)+o(1)} \{W(b)+x+o(1)\}\, d\psi_n(b;x) - \int_0^{-W(a)+o(1)} \{W(a)+x+o(1)\}\, d\psi_n(a;x) \right|$$

almost surely as $n \to \infty$. Hence, introducing the event $A = \{-1 < -W(b) < -1/2,\ 1/2 < -W(a) < 1\}$ and using (2.8) and (2.9), we see that $D_n(a,b)\, 1\!\!1_A \le 1\!\!1_A\{4 + o(1)\}$ almost surely for all n large enough, where $1\!\!1_A$ is the indicator of A. Noting that $P\{A\} > 0$, the lemma follows. □

LEMMA 5. *Whenever there exist a subsequence* $\{n'\} \subset \{n\}$ *such that* $\Delta_{n'}(a,b) > 0$ *for all* n' *large enough and*

(2.11) $$-\psi_{n'}(a;x) \to \infty \qquad \text{for some} \quad x < 0 \quad \text{as} \quad n' \to \infty$$

or

(2.12) $$\psi_{n'}(b;x) \to \infty \qquad \text{for some} \quad x > 0 \quad \text{as} \quad n' \to \infty$$

and a sequence of positive constants $A_{n'}$, *such that*

(2.13) $$D_{n'}(a,b)\sqrt{k_{n'}}\,\Delta_{n'}(a,b)\,/\,(\sqrt{k_{n'}}\,\Delta_{n'}(a,b) \vee A_{n'}) = O_P(1)$$

as $n' \to \infty$, *where* $D_n(a,b)$ *is as in Lemma 4 and* $x \vee y = \max(x,y)$, *then*

(2.14) $$\sqrt{k_{n'}}\,\Delta_{n'}(a,b)/A_{n'} \to 0 \quad \textit{as} \quad n' \to \infty$$

and for all $x \in \mathbb{R}$,

(2.15) $$\limsup_{n'\to\infty} \sqrt{k_{n'}}\,\Delta^*_{n'}(a,b)\{|\psi_{n'}(a;x)| + |\psi_{n'}(b;x)|\}/A_{n'} < \infty.$$

Proof. First assume that (2.11) holds for some $x < 0$. (By (2.8) we only have to consider negative arguments.) Choose any $y < x < 0$ and consider the event $B = \{-W(a) < 2y,\ -1 \le -W(b) < -1/2\}$. Using (2.10), (2.9) and (2.11), we see that with probability 1 for all n' large enough,

$$D_n(a,b)\mathbb{1}_B \ge \mathbb{1}_B\left\{\int_y^0 (-2y + z + o(1))\, d\psi_{n'}(a;z) - 2(1+o(1))\right\}$$

$$\ge \mathbb{1}_B\{(y+o(1))\,\psi_{n'}(a;y) - 2(1+o(1))\}.$$

Since $P\{B\} > 0$ and $y\,\psi_{n'}(a;y) \to \infty$ as $n' \to \infty$, in order for (2.13) to hold we obviously must have

$$\limsup_{n'\to\infty} -\psi_{n'}(a;x)\sqrt{k_{n'}}\,\Delta_{n'}(a,b)/(\sqrt{k_{n'}}\,\Delta_{n'}(a,b) \vee A_{n'}) < \infty$$

for all $x \in \mathbb{R}$ by the monotonicity of $\psi_{n'}(a;\cdot)$, which of course can only happen if (2.14) holds and

$$\limsup_{n'\to\infty} |\psi_{n'}(a;x)|\sqrt{k_{n'}}\,\Delta_{n'}(a,b)/A_{n'} < \infty \qquad \text{for all} \quad x \in \mathbb{R}.$$

A similar argument shows that if (2.12) holds then again (2.14) must be true and

$$\limsup_{n'\to\infty} |\psi_{n'}(b;x)|\sqrt{k_{n'}}\,\Delta_{n'}(a,b)/A_{n'} < \infty \qquad \text{for all} \quad x \in \mathbb{R}.$$

Putting everything together and noting also that by (1.6) we presently have

$\Delta_{n'}(a,b) = \Delta^{*}_{n'}(a,b)$ for all large n', it is now routine to argue that whenever (2.11) or (2.12) hold along with (2.13), we must have (2.14) and (2.15). □

A slight variation of the proof of Lemma 5 also gives the following.

LEMMA 6. *Whenever there exist a subsequence* $\{n'\} \subset \{n\}$ *such that* $\Delta_{n'}(a,b) = 0$ *for every* n' *and positive constants* $A_{n'}$ *such that* $\{R_{n'}(b)-R_{n'}(a)\}/A_{n'} = O_P(1)$ *as* $n' \to \infty$, *then we have (2.15) for all* $x \in \mathbb{R}$.

Next we look at the term M_n in (2.6). Since, using the definition in (1.5) and the notation $x \wedge y = \min(x,y)$, we clearly have

$$EM_n^2 = \int_{\lceil ak_n \rceil/n}^{\lceil bk_n \rceil/n} \int_{\lceil ak_n \rceil/n}^{\lceil bk_n \rceil/n} n(u \wedge v - uv)\, dH(u)dH(v) \le \lceil bk_n \rceil\, \Delta_n^2(a,b)$$

for any $n \ge 1$, we obtain the following.

LEMMA 7. *On any subsequence* $\{n'\} \subset \{n\}$ *on which* $\Delta_{n'}(a,b) > 0$ *for all* n' *large enough,*

$$M_{n'}/(\sqrt{k_{n'}}\, \Delta^{*}_{n'}(a,b)) = O_P(1) \quad \text{as} \quad n' \to \infty.$$

Finally we quote a variant of Lemma 2.10 of Csörgő, Haeusler, and Mason (1988), which is obtained by the original proof.

LEMMA 8. *Let* $Y_{1,n}$ *and* $Y_{2,n}$ *be two sequences of random variables such that* $Y_{1,n} + Y_{2,n} = O_P(1)$ *as* $n \to \infty$, *the sequences* $|Y_{1,n}|$ *and* $|Y_{2,n}|$ *are asymptotically independent, and for at least one of* i = 1 *or* i = 2,

$$\lim_{M\to\infty} \liminf_{n\to\infty} P\{|Y_{i,n}| < M\} > 0. \tag{2.16}$$

Then both sequences $Y_{1,n}$ *and* $Y_{2,n}$ *are stochastically bounded.*

Proof of Theorem 2. Assume (1.16). To get rid of the centering, for each n' let $M'_{n'} + R'_{n'}(b) - R'_{n'}(a)$ be an independent copy of $M_{n'} + R_{n'}(b) - R_{n'}(a)$ in the representation (2.7). Then, since (1.16) can be written as

$$(2.17) \qquad A_{n'}^{-1}\{I_{n'}(a,b) - \mu_{n'}(a,b)\} + A_{n'}^{-1}(\mu_{n'}(a,b) - C_{n'}) \to_{\mathfrak{D}} V^*$$

as $n' \to \infty$, it implies that the sequence

$$\{M_{n'} + R_{n'}(b) - R_{n'}(a) - (M'_{n'} + R'_{n'}(b) - R'_{n'}(a))\} / (A_{n'} \vee \sqrt{k_{n'}}\, \Delta_{n'}(a,b))$$

is stochastically bounded. Since by Lemmas 4 and 7 the sequence $Y_{1,n'} = \{M_{n'} + R_{n'}(b) - R_{n'}(a)\} / (A_{n'} \vee \sqrt{k_{n'}}\, \Delta_{n'}(a,b))$ obviously satisfies (2.16), Lemma 8 therefore forces

$$\{M_{n'} + R_{n'}(b) - R_{n'}(a)\}/(A_{n'} \vee \sqrt{k_{n'}}\, \Delta_{n'}(a,b)) = O_P(1) \quad \text{as } n' \to \infty,$$

which by Lemma 7 implies that

$$(2.18) \quad \{R_{n'}(b) - R_{n'}(a)\}/(A_{n'} \vee \sqrt{k_{n'}}\, \Delta_{n'}(a,b)) = O_P(1) \qquad \text{as } n' \to \infty.$$

We shall now show that (2.17) and (2.18) imply the existence of a subsequence $\{n''\} \subset \{n'\}$ such that (1.17) and (1.18) hold along $\{n''\}$. We must consider three separate cases.

Case 1: $\Delta_{n'}(a,b) > 0$ *for all* n' *large enough and*

$$(2.19) \quad \limsup_{n'\to\infty} \left(|\psi_{n'}(a;x)| + |\psi_{n'}(b;x)|\right) < \infty \quad \textit{for all } x \in \mathbb{R}.$$

In this case, by a Helly selection one can choose a subsequence $\{n'''\} \subset \{n'\}$ such that

$$(2.20) \quad \psi_{n'''}(c;\cdot) \Rightarrow \psi_c^*(\cdot) \quad \text{on } \mathbb{R} \quad \text{as } n''' \to \infty, \quad c = a,b,$$

for some functions ψ_a^* and ψ_b^* satisfying the usual conditions. By Lemma 1 and its proof we know that for a function φ, along perhaps a further subsequence $\{n''\} \subset \{n'''\}$ we have (1.9),

$$Y_{n''} := \frac{M_{n''}}{\sqrt{k_{n''}}\ \Delta^{*}_{n''}(a,b)} \to Z(\varphi) := \int_{[a,b]} W(s)d\varphi(s)$$

and

$$Y_{n''} = \int_a^{\lceil bk_{n''}\rceil/k_{n''}} W(s)d\varphi_{n''}(s) + o(1) =: Z_{n''} + o(1)$$

almost surely as $n'' \to \infty$, where $Z_{n''}$ is easily seen to be a normal random variable with mean zero and

$$E\, Z^2_{n''} = \int_a^{\lceil bk_{n''}\rceil/k_{n''}} \int_a^{\lceil bk_{n''}\rceil/k_{n''}} (u \wedge v)d\varphi_{n''}(u)d\varphi_{n''}(v) \geq a.$$

This implies that the normal random variable $Z(\varphi)$ is non-degenerate, which in turn implies that $\varphi \not\equiv 0$. Then by part (i) of Theorem 1 we have (1.10) with the non-degenerate limit $V(\psi^{*}_a, \varphi, \psi^{*}_b)$ and this, (2.17), and the convergence of types theorem yield (1.18) and

(2.21) $\{\mu_{n''}(a,b) - C_{n''}\}/A_{n''} \to \gamma$ as $n'' \to \infty$

for some constant $\gamma \in \mathbb{R}$. Now (1.18) and (2.20) imply (1.17) with $\psi_c(\cdot) = \delta\, \psi^{*}_c(\cdot)$, c=a,b, and we see that for V^{*} in (1.16) we have (1.19) with the γ from (2.21).

Case 2: $\Delta_{n'}(a,b) > 0$ *for all* n' *large enough and we have* (2.11) *or* (2.12). In this case (2.18) means exactly condition (2.13) of Lemma 5 and hence we have (2.14) and (2.15). Therefore, by (2.15) and a Helly selection we see that there exist an $\{n''\} \subset \{n'\}$ and two functions ψ_a and ψ_b with the usual properties such that (1.17) holds. Now (2.14) means that we have (1.18) with $\delta = 0$ along the original $\{n'\}$ in fact. Thus by part (ii) of Theorem 1,

$$\{I_{n''}(a,b) - \mu_{n''}(a,b)\}/A_{n''} \to_{\mathcal{D}} V(\psi_a, 0, \psi_b) \quad \text{as} \quad n'' \to \infty,$$

which in conjunction with (2.17) gives (2.21) for some $\gamma \in \mathbb{R}$ by convergence of types. Whence we also have (1.19) with $\delta=0$.

Case 3: *There exists a subsequence* $\{n'''\} \subset \{n'\}$ *such that* $\Delta_{n'''}(a,b) = 0$ *for all* n'''. Then by (2.18) and Lemma 6 there exist an $\{n''\} \subset \{n'''\}$ and two functions ψ_a and ψ_b with the usual properties such that (1.17) holds, and since (1.18) now trivially holds with $\delta=0$, the proof can be finished exactly as in Case 2. This completes the proof of the theorem. □

Proof of Theorem 3. First we assume (I). Let k_n be an arbitrary positive sequence satisfying (1.1) and choose any subsequence $\{n'\} \subset \{n\}$. Then there exist a subsequence $\{n''\} \subset \{n'\}$ and $0 < a_o < b_o < \infty$ such that for any choice of $0 < a < a_o < b_o < b < \infty$ and $\{n'''\} \subset \{n''\}$ there is a further subsequence $\{n^{iv}\} \subset \{n'''\}$ such that (1.21) holds. This implies by Theorem 2 that for some $\{n^v\} \subset \{n^{iv}\}$,

$$\bar{\psi}_{n^v}(c;\cdot) = \frac{\Delta^*_{n^v}(a,b)}{B\left(\left\lfloor n^v/k_{n^v}\right\rfloor\right)}\psi_{n^v}(c;\cdot) \Rightarrow \psi_c(\cdot) \quad \text{on } \mathbb{R}, \quad c=a,b,$$

and

$$\Delta_{n^v}(a,b)/B\left(\left\lfloor n^v/k_{n^v}\right\rfloor\right) \to \delta(a,b) \quad \text{a } n^v \to \infty, \tag{2.22}$$

where $0 \le \delta(a,b) < \infty$ and ψ_a and ψ_b are some functions satisfying the properties listed above (1.17).

Clearly, for any $a < a^* < a_o < b_o < b^* < b$ we can find a further subsequence $\{n^{vi}\} \subset \{n^v\}$ such that

$$\Delta_{n^{vi}}(a^*,b^*)/B\left(\left\lfloor n^{vi}/k_{n^{vi}}\right\rfloor\right) \to \delta(a^*,b^*) \quad \text{as} \quad n^{vi} \to \infty,$$

where $0 \le \delta(a^*,b^*) < \infty$. Noticing that for any $x \in \mathbb{R}$,

$$|\bar{\psi}_{n^{vi}}(d;x)| \le \Delta_{n^{vi}}(a,b)/B\left(\left\lfloor n^{vi}/k_{n^{vi}}\right\rfloor\right), \quad d = a^*,\ b^*,$$

for all n^{vi} sufficiently large, we see by Theorem 1 that if $\delta(a,b) = 0$, then

$$\{I_{n^{vi}}(a^*,b^*) - \mu_{n^{vi}}(a^*,b^*)\}/B\left[\lfloor n^{vi}/k_{n^{vi}}\rfloor\right] \to_P 0 \quad \text{as} \quad n^{vi} \to \infty,$$

which contradicts (I). Thus $\delta(a,b) > 0$ in (2.22).

A diagonal selection procedure now shows that a subsequence $\{n^{vii}\} \subset \{n'\}$ can be chosen so that (1.20) holds true along $\{n^{vii}\}$ with some finite function $h(\cdot)$ which is not identically zero on $(0,\infty)$, that is the de Haan-Resnick condition for (II) is satisfied.

Now assume (II) and choose any sequence $\{k_n\}$ satisfying (1.1). Let $\{n'\} \subset \{n\}$ be an arbitrary subsequence. Then there exist a further subsequence $\{n''\} \subset \{n'\}$ and a finite function $h(\cdot)$ on $(0,\infty)$ such that $h \not\equiv 0$ and we have (1.20) for some $\{B(n)\}$. Thus we can find $0 < a_o < b_o < \infty$ such that

$$\left\{h_{k_{n''}/n''}(b_o) - h_{k_{n''}/n''}(a_o)\right\} / B(\lfloor n''/k_{n''}\rfloor) \to h(b_o) - h(a_o) > 0$$

as $n'' \to \infty$. This implies that $\Delta_{n''}(a,b) > 0$ for all $0 < a < a_o < b_o < b < \infty$ and all n'' large enough. Fix $0 < a < a_o$ and $b_o < b < \infty$, and let $\{n'''\}$ be an arbitrary subsequence of $\{n''\}$. Then we can find a subsequence $\{n^v\} \subset \{n'''\}$ such that

$$\text{(2.23)} \quad \Delta_{n^v}(a,b)/B\left[\lfloor n^v/k_{n^v}\rfloor\right] \to \delta, \quad 0 < \delta < \infty, \quad \text{as } n^v \to \infty$$

and we also see that for any $x \in \mathbb{R}$, $0 < \bar{a} < a$, and $b < \bar{b} < \infty$,

$$|\psi_{n''}(c;x)| \le \left\{h_{k_{n''}/n''}(\bar{b}) - h_{k_{n''}/n''}(\bar{a})\right\} / \Delta_{n''}(a,b), \quad c=a,b,$$

for all n'' large enough. Hence we can choose a further subsequence $\{n^{vi}\} \subset \{n^v\}$ such that (1.8) holds along $\{n^{vi}\}$ with some functions ψ_a and ψ_b. But then, since $\Delta_{n^{vi}}(a,b) > 0$ for all n^{vi} large enough, there exists a final subsequence $\{n^{iv}\} \subset \{n^{vi}\}$ such that by part (i) of Theorem 1 we have (1.9) along $\{n^{iv}\}$ with some φ satisfying $\varphi(b+)=1$ and also (1.10) along $\{n^{iv}\}$. Since $\varphi \not\equiv 0$, by (2.23) we finally obtain (1.21) with the non-degenerate random variable $Y = \delta\, V(\psi_a, \varphi, \psi_b)$. □

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