# ON HADAMARD DIFFERENTIABILITY OF STATISTICAL FUNCTIONAL PROCESSES

JIAN-JIAN REN AND PRANAB KUMER SEN
University of North Carolina at Chapel Hill

Robust (M-) estimation in linear models generally involves statistical functional processes. For drawing statistical conclusions (in large samples), some (uniform) linear functional approximations are usually needed for such functionals. In this context, the role of Hadamard differentiability is critically examined.

## 1. INTRODUCTION

In nonparametric models, a parameter $\theta$ (= $T(F)$) is regarded as a functional $T(\cdot)$ on a space $\mathcal{F}$ of distribution functions (d.f.) F. Thus, the same functional of the sample d.f. $F_n$ (i.e., $T(F_n)$) is regarded as a natural estimator of $\theta$. Using a form of the Taylor expansion involving the derivatives of the functional, von Mises [7] expressed $T(F_n)$ as

$$T(F_n) = T(F) + T_F'(F_n-F) + \text{Rem}(F_n-F;\ T(\cdot)) \tag{1.1}$$

AMS (1980) Subject Classifications: 62F35, 62G99

Key words and phrases: asymptotic linearity, Fréchet differentiability, Gâteaux differentiability, Hadamard differentiability, influence function, robust (M-) estimation, statistical functional, statistical functional process.

where $T'_F$ is the derivative of the functional at F and $\mathrm{Rem}(F_n-F;\ T(\cdot))$ is the remainder term in this first order expansion. Note that $F_n(x)$ $(= n^{-1}\ \Sigma_{i=1}^{n}\ I(X_i \leq x))$ is based on n independent and identically distributed random variables (i.i.d.r.v.) $X_1,\ldots,X_n$, each having the d.f. F, and $T'_F(F_n-F)$ is a linear functional. Hence, $T'_F(F_n-F)$ is an average of n i.i.d.r.v.'s. For drawing statistical conclusions (in large samples), $T'_F$ plays the basic role, and in this context, it remains to show that $\mathrm{Rem}(F_n-F;\ T(\cdot))$ is negligible to the desired extent. Appropriate differentiability conditions are usually incorporated towards this verification.

We also observe that a statistical functional induces a functional on the space D[0,1] (of right continuous functions having left hand limits) in the following way:

$$\tau(G) = T(G \circ F), \quad G \in D[0,1]. \tag{1.2}$$

Thus, (1.1) can be written equivalently as

$$\tau(U_n) = \tau(U) + \tau'_U(U_n-U) + \mathrm{Rem}(U_n-U;\ \tau(\cdot)), \tag{1.3}$$

where $U_n$ is the empirical d.f. of the $Y_i$ $(= F(X_i))$, $1 \leq i \leq n$, and U is the classical uniform d.f. on [0,1] (i.e., $U(t) = t$, $0 \leq t \leq 1$). Since the expansion in (1.1), rewritten in (1.3), is based on some kind of differentiation, it is quite natural to inquire about the right form of such a differentiation to suit the desired purpose. The current literature is based on an extensive use of the Fréchet derivatives which are generally too stringent. Less restrictive concepts envolve the Gâteaux and Hadamard (or compact) derivatives (viz., Kallianpur [4], Reeds [6] and Fernholz [2], among others). Using the Hadamard differentiability (along with some other regularity conditions), Reeds [6] has shown that

$$n^{1/2}\{\mathrm{Rem}(U_n-U;\ \tau(\cdot))\} \xrightarrow{P} 0, \quad \text{as} \quad n \to \infty \tag{1.4}$$

so that noting that $\tau_U'(U_n-U) = n^{-1}\ \Sigma_{i=1}^n\ IC(X_i;F,T)$, where $IC(x;F,T)$ is the influence function of T at F, and assuming that $\sigma^2 = \mathrm{Var}_F\ IC(X_i;F,T) < \infty$, one obtains that

$$n^{1/2}(T(F_n)-\theta) \overset{P}{\sim} n^{1/2}\ \tau_U'(U_n-U) \xrightarrow{\mathfrak{D}} N(0,\sigma^2) \tag{1.5}$$

In the context of the law of iterated logarithm or some almost sure (a.s.) representation for $T(F_n)$, one may require a stronger mode of convergence in (1.4), and this, in turn, may require a more stringent differentiability condition. However, in a majority of statistical applications, Hadamard differentiability suffices, and we shall explore this concept in the context of functional process arising in roubst (M-) estimation in simple linear models.

Consider the simple linear model:

$$X_i = \underset{\sim}{\beta}'\underset{\sim}{c}_i + e_i, \quad i \geq 1. \tag{1.6}$$

where the $\underset{\sim}{c}_i$ are known p-vectors of regression constants, $\underset{\sim}{\beta} = (\beta_1,\ldots,\beta_p)'$ is the vector of unknown (regression) parameters, $p \geq 1$, and $e_i$ are i.i.d.r.v. with d.f. $F(\in \mathcal{F})$. Based on a suitable score function $\psi : R \to R$, an M-estimator $\hat{\underset{\sim}{\beta}}_n$ of $\underset{\sim}{\beta}$ is defined as a solution (with respect to $\underset{\sim}{\theta}$) of the following equations:

$$\Sigma_{i=1}^n\ \underset{\sim}{c}_i\ \psi(X_i-\underset{\sim}{\theta}'\underset{\sim}{c}_i) \equiv \underset{\sim}{0}, \tag{1.7}$$

where "$\equiv 0$" accommodates the possibility of left hand side being closest to $\underset{\sim}{0}$ when equality in (1.7) is unattainable (such a case may arise when $\psi$ is not continuous everywhere). Setting $Z_i = X_i-\underset{\sim}{\beta}'\underset{\sim}{c}_i$ (i.i.d. with d.f. F), we shall see that the following empirical function

$$\underset{\sim}{S}_n^*(t,\underset{\sim}{u}) = \Sigma_{i=1}^n \underset{\sim}{c}_{ni} \; I(Z_i \le F^{-1}(t) + \underset{\sim}{u}'\underset{\sim}{c}_{ni}), \quad t \in [0,1], \; \underset{\sim}{u} \in R^p, \tag{1.8}$$

arises typically in the study of the asymptotic properties of $\hat{\underset{\sim}{\beta}}_n$, where the $\underset{\sim}{c}_{ni}$ are suitably normalized version of the $\underset{\sim}{c}_i$. For example, we may set for every $n \ge p$,

$$\underset{\sim}{C}_n = \Sigma_{i=1}^n \underset{\sim}{c}_i \underset{\sim}{c}_i' = (c_{njj'})_{1\le j,j'\le p}, \tag{1.9}$$

$$\underset{\sim}{C}_n^o = \mathrm{Diag}(c_{n11}^{1/2},\dots,c_{npp}^{1/2}), \tag{1.10}$$

$$\underset{\sim}{c}_{ni} = (\underset{\sim}{C}_n^o)^{-1} \underset{\sim}{c}_i, \quad i = 1,\dots,n. \tag{1.11}$$

Thus, letting

$$\underset{\sim}{u} = \underset{\sim}{C}_n^o(\underset{\sim}{\theta} - \underset{\sim}{\beta}) \tag{1.12}$$

and

$$\underset{\sim}{M}_n(\underset{\sim}{u}) = \Sigma_{i=1}^n \underset{\sim}{c}_{ni} \; \psi(Z_i - \underset{\sim}{u}'\underset{\sim}{c}_{ni}), \tag{1.13}$$

we see that (1.7) is equivalent to

$$\underset{\sim}{M}_n(\underset{\sim}{u}) \equiv \underset{\sim}{0} \qquad \text{(with respect to } \underset{\sim}{u}). \tag{1.14}$$

The solution of this implicit (set of) equations is greatly facilitated by the following type of (Jurecková-) linearity for M-processes: For every finite $K : 0 < K < \infty$, as $n \to \infty$,

$$\sup\{\|\underset{\sim}{M}_n(\underset{\sim}{u}) - \underset{\sim}{M}_n(\underset{\sim}{0}) + \underset{\sim}{Q}_n \underset{\sim}{u} \gamma\|; \; \|\underset{\sim}{u}\| \le K\} \overset{P}{\to} 0, \tag{1.15}$$

where $\|\cdot\|$ stands for the Euclidean norm, $\gamma = \int \psi' dF$ and $\underset{\sim}{Q}_n = \Sigma_{i=1}^n \underset{\sim}{c}_{ni} \underset{\sim}{c}_{ni}' = (\underset{\sim}{C}_n^o)^{-1} \underset{\sim}{C}_n (\underset{\sim}{C}_n^o)^{-1}$. Under various conditions on the $\underset{\sim}{c}_{ni}$, the score function $\psi$ and the d.f. F, (1.15) has been established (Jurecková [3]), and this provides an easy access to the study of the asymptotic properties of the M-estimator $\hat{\underset{\sim}{\beta}}_n$.

We consider a different approach here. For simplicity of presentation, we consider explicitly the case of p=1, i.e., the $c_{ni}$ are real numbers. We consider a statistical functional process $\left\{\tau\left[\frac{S_n^*(\cdot,u)}{\Sigma_{j=1}^n c_{nj}}\right]: u \in R\right\}$, where $\tau$ is given in (1.2), and for this statistical functional process, the results corresponding to (1.4) are shown in this paper. Since for each $u \in R$, $M_n(u)$ is a linear functional of $S_n^*(t,u) = \sum_{i=1}^n c_{ni}\ I(Z_i \le F^{-1}(t) + uc_{ni})$, viz.,

$$M_n(u) = \int \psi(F^{-1}(t))\ d\ S_n^*(t,u) \tag{1.16}$$

$M_n(u)$ could be the Hadamard derivative of a certain functional $\tau$. Thus, using the results of this paper, for a proper functional $\tau$, we have, for any $K > 0$,

$$\sup_{|u|\le K}\left|\sum_{i=1}^n c_{ni}\left[\tau\left[\frac{S_n^*(\cdot,u)}{\Sigma_{i=1}^n c_{ni}}\right] - \tau\left[\frac{S_n^*(\cdot,0)}{\Sigma_{i=1}^n c_{ni}}\right]\right] - [M_n(u) - M_n(0)]\right| \xrightarrow{P} 0. \tag{1.17}$$

Therefore, (1.15) follows from showing

$$\sup_{|u|\le K}\left|\sum_{i=1}^n c_{ni}\left[\tau\left[\frac{S_n^*(\cdot,u)}{\Sigma_{i=1}^n c_{ni}}\right] - \tau\left[\frac{S_n^*(\cdot,0)}{\Sigma_{i=1}^n c_{ni}}\right]\right] + u\gamma\right| \xrightarrow{P} 0. \tag{1.18}$$

Some notation along with the basic assumptions are presented in Section 2. In the same section, the notion of statistical functionals and the concept of Hadamard differentiability are also introduced. The main results along with part of their derivations are considered in Section 3. The proof of Theorem 3.1 is given separately in Section 4.

## 2. PRELIMINARY NOTIONS

Consider the D[0,1] space (of right continuous real valued functions with left limits) endowed with the uniform topology†. The space C[0,1] of real valued continuous functions, endowed with the uniform topology, is a subspace of D[0,1]. For every $u \in R$, denote by

$$S_n^*(t,u) = \Sigma_{i=1}^n c_{ni} \; I(Z_i \le F^{-1}(t) + uc_{ni}), \quad t \in [0,1], \tag{2.1}$$

where the $c_{ni}$ are all given real numbers, and the $Z_i$ are i.i.d.r.v. with the d.f. F. It is easy to see that for every u ($\in R$), $S_n^*(t;u)$ is an element of D[0,1]. The population counterpart of the $I(Z_i \le F^{-1}(t) + uc_{ni})$ are the $F(F^{-1}(t) + uc_{ni})$, and this leads us to consider the following:

$$S_n(t,u) = \Sigma_{i=1}^n c_{ni} \; F(F^{-1}(t) + u\,c_{ni}), \qquad t \in [0,1], \quad u \in R. \tag{2.2}$$

We also write for every $i \ge 1$,

$$c_{ni} = c_{ni}^+ - c_{ni}^-; \quad c_{ni}^+ = \max\{0,\ c_{ni}\},\ c_{ni}^- = -\min\{0, c_{ni}\}; \tag{2.3}$$

$$S_n^{*+}(t,u) = \Sigma_{i=1}^n c_{ni}^+ \; I(Z_i \le F^{-1}(t) + u\,c_{ni}^+) \tag{2.4}$$

$$S_n^{*-}(t,u) = \Sigma_{i=1}^n c_{ni}^- \; I(Z_i \le F^{-1}(t) - u\,c_{ni}^-) \tag{2.5}$$

So that

$$S_n^*(t,u) = S_n^{*+}(t,u) - S_n^{*-}(t,u), \quad t \in [0,1], \quad u \in R. \tag{2.6}$$

Let then

$$W_n(t,u) = S_n^*(t,(2u-1)K) - S_n(t,(2u-1)K), \quad (t,u) \in [0,1]^2. \tag{2.7}$$

---

†Although D[0,1] is endowed with the Skorohod-$J_1$ topology, in view of our uniformity result in (1.15), we shall use the uniform topology.

where K is a positive real number, and let

$$W_n^o(t,u) = S_n(t, (2u-1)K) - t\,\Sigma_{i=1}^n c_{ni}; \quad (t,u) \in [0,1]^2. \tag{2.8}$$

We also denote by

$$(d_n^+)^2 = \Sigma_{i=1}^n (c_{ni}^+)^2, \text{ and } (d_n^-)^2 = \Sigma_{i=1}^n (c_{ni}^-)^2. \tag{2.9}$$

and for any function f on $[0,1]^2$,

$$\omega_f(\delta) = \sup\{|f(t,u) - f(s,v)|; \ |t-s| \le \delta, \ |u-v| \le \delta\} \tag{2.10}$$

Some assumptions, which may be required for our main results, are given below:

(A1) $c_{ni} \geq 0$

(A2) $\Sigma_{i=1}^n c_{ni}^2 = 1, \quad \overline{\lim_{n\to\infty}}\, n \max_{1\le i\le n} c_{ni}^2 < \infty$

(B) F is absolutely continuous with density function F' which is positive and continuous with limits at ± ∞.

In order to prove our main results in Section 3, some basic concepts about statistical functional and Hadamard differentiability are needed.

DEFINITION. Let $X_1,\ldots,X_n$ be a sample from a population with d.f. F and let $T_n = T_n(X_1,\ldots,X_n)$ be a statistics. If $T_n$ can be written as a functional T of the empirical d.f. $F_n$ corresponding to the sample $X_1,\ldots,X_n$, i.e., $T_n = T(F_n)$, where T does not depend on n, then T will be called a *statistical functional*.

We actually consider an extended statistical functional, or a statistical functional process: $T_{n,u} = T\left[\dfrac{S_n^*(F(\cdot),u)}{\Sigma_{i=1}^n c_{ni}}\right]$, $u \in R$. The domain of definition of T is assumed to contain $S_n^*(F(\cdot),u)/\Sigma_{i=1}^n c_{ni}$ for all $n \geq 1$

and $u \in R$, as well as the population d.f. F. Usually, the range of T will be the set of real numbers.

Any statistical functional T induces a functional $\tau$ on D[0,1] by the relation in (1.2). In Section 3 and Section 4, we will always assume functional $\tau$ is induced by a statistical functional T.

<u>DEFINITION</u>. The *influence function* (IF) or *influence curve* (IC) of a statistics T at a d.f. F is usually defined by

$$IC(x;F,T) = \frac{d}{dt} T(F + t(\delta_x - F))\big|_{t=0}$$

where $\delta_x$ is the d.f. of the point mass one at x.

Let V and W be the topological vector spaces and L(V,W) be the set of continuous linear transformation from V to W. Let $\mathscr{A}$ be an open subset of V,

<u>DEFINITION</u>. A functional $T : \mathscr{A} \to W$ is *Hadamard-differentiable* (or compact differentiable) at $F \in \mathscr{A}$ if there exists $T_F' \in L(V,W)$ such that for any compact set $\Gamma$ of V,

$$\lim_{t \to 0} \frac{T(F+tH) - T(F) - T_F'(tH)}{t} = 0 \tag{2.11}$$

uniformly for any $H \in \Gamma$. The linear function $T_F'$ is called the *Hadamard derivative* of T at F.

For convenience sake, in (2.11) we usually denote

$$Rem(tH) = T(F + tH) - T(F) - T_F'(tH) \tag{2.12}$$

then, correspondently in Section 3 and Section 4, we always use

$$Rem(tH) = \tau(U + tH) - \tau(U) - \tau_U'(tH). \tag{2.13}$$

By the two previous definitions, we can easily see that the existance of Hadamard derivative implies the existance of the influence curve and we also have

$$T_F'(\delta_x - F) = IC(x;F,T) \tag{2.14}$$

## 3. MAIN RESULTS

THEOREM 3.1. Suppose $\tau : D[0,1] \to R$ is a functional and is Hadamard differentiable at U. Assume (A1), (A2) and (B). Then, for any $K > 0$,

$$\sup_{|u|\le K} \left| \sum_{i=1}^{n} c_{ni} \operatorname{Rem}\left[ \frac{S_n^*(\cdot,u)}{\Sigma_{i=1}^n c_{ni}} - U(\cdot) \right] \right| \xrightarrow{P} 0, \quad \text{as} \quad n \to \infty. \tag{3.1}$$

Therefore, we have, as $n \to \infty$,

$$\sup_{|u|\le K} \left| \sum_{i=1}^{n} c_{ni} \left[ \tau\left[ \frac{S_n^*(\cdot,u)}{\Sigma_{i=1}^n c_{ni}} \right] - \tau(U(\cdot)) \right] - \tau_U'(S_n^*(\cdot,u) - U(\cdot)\Sigma_{i=1}^n c_{ni}) \right| \xrightarrow{P} 0. \tag{3.2}$$

The proof of Theorem 3.1 will be given in Section 4. When (A1) is not satisfied, we have the following theorem.

THEOREM 3.2. Suppose $\tau$ $D[0,1] \to R$ is a functional and is Hadamard differentiable at U. Assume (A2) and (B), and assume $\overline{\lim}_{n\to\infty} n \max_{1\le i\le n} (c_{ni}^+)^2/(d_n^+)^2 < \infty$, $\overline{\lim}_{n\to\infty} n \max_{1\le i\le n} (c_{ni}^-)^2/(d_n^-)^2 < \infty$. Then, for any $K > 0$, as $n \to \infty$

$$\sup_{|u|\le K} \left| \sum_{i=1}^{n} c_{ni}^+ \, \tau\left[ \frac{S_n^{*+}(\cdot,u)}{\Sigma_{i=1}^n c_{ni}^+} \right] - \sum_{i=1}^{n} c_{ni}^- \, \tau\left[ \frac{S_n^{*-}(\cdot,u)}{\Sigma_{i=1}^n c_{ni}^-} \right] \right.$$
$$\left. - \tau(U)\, \Sigma_{i=1}^n c_{ni} - \tau_U'(S_n^*(\cdot,u) - U(\cdot) \sum_{i=1}^{n} c_{ni}) \right| \xrightarrow{P} 0. \tag{3.3}$$

Proof. Denote $\bar{c}_{ni}^+ = c_{ni}^+/d_n^+$ and $\bar{c}_{ni}^- = c_{ni}^-/d_n^-$. By the assumptions on $c_{ni}^+$ and $c_{ni}^-$, we have

$$\overline{\lim}_{n\to\infty} n \max_{1\le i\le n} (\bar{c}_{ni}^+)^2 < \infty, \quad \text{and} \quad \overline{\lim}_{n\to\infty} n \max_{1\le i\le n} (\bar{c}_{ni}^-)^2 = < \infty.$$

Note $\sum_{i=1}^{n} (\bar{c}_{ni}^{+})^2 = 1$, $\Sigma_{i=1}^{n} (\bar{c}_{ni}^{-})^2 = 1$ and $\Sigma_{i=1}^{n} c_{ni}^2 = (d_n^+)^2 + (d_n^-)^2 = 1$,

hence, for $\bar{c}_{ni}^{+}$ and $\bar{c}_{ni}^{-}$, (A1) and (A2) are satisfied, and

$0 < (d_n^+)^2, (d_n^-)^2 < 1$.

We also observe, for $|u| \le K$,

$$\frac{S_n^{*+}(t,u)}{\Sigma_{i=1}^{n} c_{ni}^{+}} = \frac{\Sigma_{i=1}^{n} \bar{c}_{ni}^{+} I(Z_i \le F^{-1}(t) + c_{ni}^{+} u)}{\Sigma_{i=1}^{n} \bar{c}_{ni}^{+}}$$

$$= \frac{\Sigma_{i=1}^{n} \bar{c}_{ni}^{+} I(Z_i \le F^{-1}(t) + \bar{c}_{ni}^{+} u_1)}{\Sigma_{i=1}^{n} \bar{c}_{ni}^{+}}, \qquad |u_1| \le K.$$

Therefore, by Theorem 3.1, we have, as $n \to \infty$,

$$\sup_{|u| \le K} \left| \sum_{i=1}^{n} \bar{c}_{ni}^{+} \left[ \tau \left( \frac{S_n^{*+}(\cdot,u)}{\Sigma_{i=1}^{n} c_{ni}^{+}} \right) - \tau(U(\cdot)) \right] - \tau_U' \left[ \frac{S_n^{*+}(\cdot,u)}{d_n^{+}} - U(\cdot) \sum_{i=1}^{n} \bar{c}_{ni}^{+} \right] \right| \xrightarrow{P} 0.$$

Since $0 < d_n^+ < 1$, we have, as $n \to \infty$,

$$\sup_{|u| \le K} \left| \sum_{i=1}^{n} c_{ni}^{+} \left[ \tau \left( \frac{S_n^{*+}(\cdot,u)}{\Sigma_{i=1}^{n} c_{ni}^{+}} \right) - \tau(U(\cdot)) \right] - \tau_U'(S_n^{*+}(\cdot,u) - U(\cdot) \sum_{i=1}^{n} c_{ni}^{+}) \right| \xrightarrow{P} 0. \tag{3.4}$$

Similarly, as $n \to \infty$,

$$\sup_{|u| \le K} \left| \sum_{i=1}^{n} c_{ni}^{-} \left[ \tau \left( \frac{S_n^{*-}(\cdot,u)}{\Sigma_{i=1}^{n} c_{ni}^{-}} \right) - \tau(U(\cdot)) \right] - \tau_U'(S_n^{*-}(\cdot,u) - U(\cdot) \sum_{i=1}^{n} c_{ni}^{-}) \right| \xrightarrow{P} 0. \tag{3.5}$$

Therefore, (3.3) follows from (3.4) and (3.5). □

COROLLARY 3.3. Suppose $\tau : D[0,1] \to R$ is Hadamard differentiable at U. Assume (A2) and (B), and assume

$$\underline{\lim}_{n\to\infty} d_n^+ > 0 \qquad \text{and} \qquad \underline{\lim}_{n\to\infty} d_n^- > 0.$$

Then, we have (3.3).

Proof. By the assumptions on $d_n^+$ and $d_n^-$, there exists a real number $c > 0$ such that $d_n^+ \geq c$ and $d_n^- \geq c$, for all n. By (A2), we have

$$\overline{\lim}_{n\to\infty} n \max_{1\leq i\leq n} (c_{ni}^+)^2/(d_n^+)^2 \leq \frac{1}{c} \overline{\lim}_{n\to\infty} n \max_{1\leq i\leq n} c_{ni}^2 < \infty$$

and

$$\overline{\lim}_{n\to\infty} n \max_{1\leq i\leq n} (c_{ni}^-)^2/(d_n^-)^2 \leq \frac{1}{c} \overline{\lim}_{n\to\infty} n \max_{1\leq i\leq n} c_{ni}^2 < \infty$$

Therefore, (3.3) follows from the proof of Theorem 2. □

## 4. PROOF OF THEOREM 3.1

In this section, (A1), (A2) and (B) are assumed. First, we notice that (A1) and (A2) imply three facts: there exists $M > 0$ such that for n large enough

$$\left[\sum_{i=1}^{n} c_{ni}\right] \max_{1\le i\le n} c_{ni} \le M, \tag{4.1}$$

$$\max_{1\le i\le n} c_{ni}^2 \to 0, \qquad \text{as } n \to \infty; \tag{4.2}$$

and further

$$\sum_{i=1}^{n} c_{ni} \to \infty, \qquad \text{as } n \to \infty. \tag{4.3}$$

The result (3.1) will first be proved on $C[0,1]$, and then be extended to $D[0,1]$.

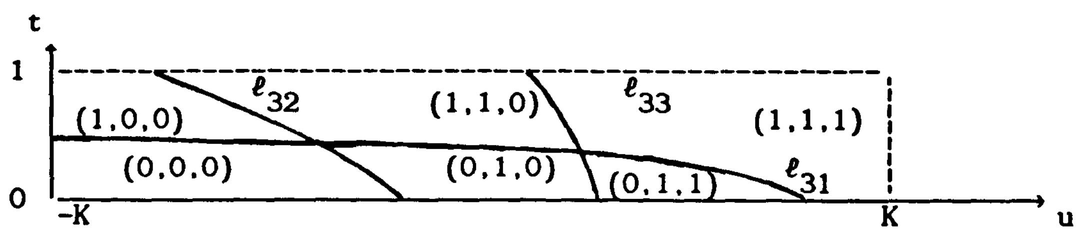
Consider

$$S_n^*(t,u) = \sum_{i=1}^{n} c_{ni}\, I(Z_i \le F^{-1}(t) + c_{ni}u), \qquad |u| \le K.$$

For each i:

$$I(Z_i \le F^{-1}(t) + c_{ni}u) = \begin{cases} 1 & \text{if } F(Z_i - c_{ni}u) \le t \\ 0 & \text{otherwise} \end{cases}$$

The curve $\ell_{ni}$ : $t = F(Z_i - c_{ni}u)$ is nonincreasing in u. Hence, for each n and $Z_1,\ldots,Z_n$, $[0,1]\times[-K,K]$ is divided into finite pieces by curves $\ell_{n1},\ldots,\ell_{nn}$, shown as the following (for n=3):



and in region (0,0,0): $S_n^*(t,u) = 0$;

region (0,1,0): $S_n^*(t,u) = c_{n2}$;

region (0,1,1): $S_n^*(t,u) = c_{n2} + c_{n3}$;

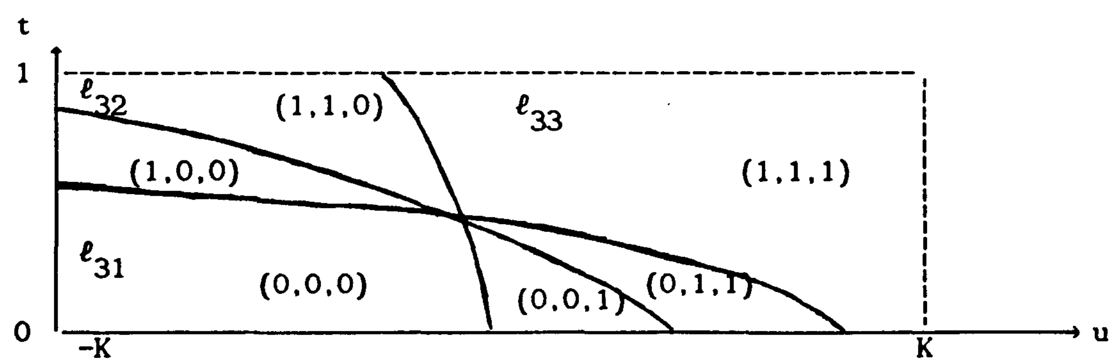
region (1,0,0): $S_n^*(t,u) = c_{n1}$;

region (1,1,0): $S_n^*(t,u) = c_{n1} + c_{n2}$;

region (1,1,1): $S_n^*(t,u) = c_{n1} + c_{n2} + c_{n3}$.

Let $\bar{S}_3^*(t,u)$ be a bivariate continuous version of $S_3^*(t,u)$. Generally, $\bar{S}_n^*(t,u)$ can be got easily by smoothing $S_n^*(t,u)$ through the above regions. Since $\bar{S}_n^*(t,u)$ is bivariate continuous, $\bar{S}_n^*(t,(2u-1)K)$ is an element of $C[0,1]^2$.

If $\ell_{31}$, $\ell_{32}$, $\ell_{33}$ intersect at one point shown as the following,




the biggest jump of $S_3^*$ is $c_{31} + c_{32} + c_{33}$. But, we will show in Lemma 4.1 that, with probability one, no more than two curves will intersect at one point in $[0,1] \times [-K,K]$.

LEMMA 4.1. For each n, no more than two $\ell$'s intersect at one point in $[0,1] \times [-K,K]$ with probability one.

Proof. Suppose $\ell_{n1}$, $\ell_{n2}$ and $\ell_{n3}$ intersect at one point $(t_o, u_o)$. Since F is increasing, we have

$$Z_1 - c_{n1}\, u_o = Z_2 - c_{n2}\, u_o = Z_3 - c_{n3}\, u_o$$

and $c_{ni} \neq c_{nj}$ for $i \neq j$ in (4.4) because, with probability one, $Z_i \neq Z_j$ for $i \neq j$. Therefore, $\ell_{n1}$, $\ell_{n2}$ and $\ell_{n3}$ intersect at one point iff

$$u_o = \frac{Z_1 - Z_2}{c_{n1} - c_{n2}} = \frac{Z_1 - Z_3}{c_{n1} - c_{n3}} .$$

Since F is continuous,

$$P\left[ \frac{Z_1 - Z_2}{c_{n1} - c_{n2}} = \frac{Z_1 - Z_3}{c_{n1} - c_{n3}} \right] = P\left[ Z_2 = \frac{c_{n2} - c_{n3}}{c_{n1} - c_{n3}} Z_1 + \frac{c_{n1} - c_{n2}}{c_{n2} - c_{n3}} Z_3 \right] = 0.$$

For each n, the probability that more than two $\ell$'s intersect at one point only depend on $c_{n1}, \ldots, c_{nn}$. Hence, with probability one, no more than two $\ell$'s intersect at one point. □

Lemma 4.1 implies that, with probability one, no more than two $\ell$'s intersect at one point among $\{\ell_{n1}, \ldots \ell_{nn} \mid n \geq 1\}$. Therefore, with probability one, the biggest jump of $S_n^*(t,u)$ is no larger than $2 \max_{1 \leq i \leq n} c_{ni}$. Since $\bar{S}_n^*(t,u)$ is a bivariate continuous smoothing of $S_n^*(t,u)$, we have

$$\|\bar{S}_n^*(\cdot,\cdot) - S_n^*(\cdot,\cdot)\| \leq 2 \max_{1 \leq i \leq n} c_{ni}, \quad \text{a.s.} \tag{4.5}$$

LEMMA 4.2. Let $G(t,u) = (2u-1)K\, F'(F^{-1}(t))$. Then,

$$\sup_{(t,u)\in[0,1]^2} |W_n^o(t,u) - G(t,u)| \to 0, \quad \text{as} \quad n \to \infty.$$

Proof. By (2.8), we notice

$$\sup_{(t,u)\in[0,1]^2} |W_n^o(t,u) - G(t,u)| = \sup\left\{\left| \sum_{i=1}^n c_{ni}[F(F^{-1}(t)+c_{ni}u)-t] - u\, F'(F^{-1}(t))\right| ;\ t \in [0,1],\ |u| \le K \right\}.$$

Since $\sum_{i=1}^n c_{ni}^2 = 1$,

$$\left| \sum_{i=1}^n c_{ni}[F(F^{-1}(t) + c_{ni}u)-t] - u\, F'(F^{-1}(t))\right|$$

$$= \left| u \sum_{i=1}^n c_{ni}^2[F'(\xi_{ni}) - F'(F^{-1}(t))]\right| \le K \sum_{i=1}^n c_{ni}^2 |F'(\xi_{ni})-F'(F^{-1}(t))|$$

where $\xi_{ni}$ is between $F^{-1}(t)$ and $F^{-1}(t) + c_{ni}u$. The proof follows from (4.2) and the uniform continuity of $F'$. □

Let $E_k = [0,1]^k$, and let

$$L_k(r) = \{r^{-1}(\ell_1,\dots,\ell_k) | \ell_i = 0,1,\dots,r,\quad i=1,\dots,k\} \tag{4.6}$$

LEMMA 4.3. (Neuhaus [5]). Let $\delta > 0$, $m(\delta) = \max\{r | \delta 2^r \le 1\}$, $m > m(\delta)$ and $f : E_k \to R$ be given. Then, for points $\underset{\sim}{t}_1$, $\underset{\sim}{t}_2 \in L_k(2^m)$ with $|\underset{\sim}{t}_1-\underset{\sim}{t}_2| \le 2^{-m(\delta)}$, the following inequality holds

$$|f(\underset{\sim}{t}_1) - f(\underset{\sim}{t}_2)| \le 4\, k \sum_{r=m(\delta)}^{m} \sum_{\mu=1}^{k} \sup|f(\underset{\sim}{j}) - f(\underset{\sim}{j} + e_\mu\, 2^{-r})|$$

where the "sup" is to be taken over all $\underset{\sim}{j} \in L_k(2^r)$ with $\underset{\sim}{j} + e_\mu\, 2^{-r} \in E_k$ ($e_\mu$ denotes the $\mu$-th unit-vector of $E_k$) and $|\underset{\sim}{t}| = |(t_1,\dots,t_k)| = \max_{1\le i\le k} |t_i|$.

PROPOSITION 4.4. For any $\epsilon > 0$, $\lim_{\delta\to 0} \overline{\lim}_{n\to\infty} P(\omega_{W_n}(\delta) \geq \epsilon) = 0$.

Proof. By (2.7), we can see that $W_n(t,u)$ is a function from $E_2$ to R. For any $(t,u)$, $(s,v) \in [0,1]^2$ with $|t-s| < \delta$, $|u-v| < \delta$, there exist $t'$, $t''$, $u'$, $u''$, $s'$, $s''$, $v'$, $v'' \in L_1(2^m)$ where m is an arbitrary positive integer, such that

$$t' \leq t \leq t'',\ u' \leq u \leq u'',\ s' \leq s \leq s'',\ \ v' \leq v \leq v''$$

and

$$\max\{(t-t'),\ (t''-t),\ (u-u'),\ (u''-u),\ (s-s'),\ (s''-s),\ (v-v'),\ (v''-v)\} \leq 2^{-m}.$$

We can write

$W_n(t,u)-W_n(s,v) =$

$$[W_n(t,u) - W_n(t',u')] + [W_n(t',u') - W_n(s'',v'')] + [W_n(s'',v'') - W_n(s,v)].$$

First, we notice that

$$|s''-t'| \leq |s''-s| + |s-t| + |t-t'| \leq \delta + 2^{-m+1}$$

and

$$|v''-u'| \leq |v''-v| + |v-u| + |u-u'| \leq \delta + 2^{-m+1}.$$

We consider $[W_n(t,u) - W_n(t',u')]$. Since

$$W_n(t,u) + S_n(t,(2u-1)K) = \Sigma_{i=1}^n c_{ni}\ I(Z_i \leq F^{-1}(t) + c_{ni}(2u-1)K)$$

is a nondecreasing function in each component of $(t,u)$, we have

$$\begin{aligned} W_n(t,u) - W_n(t',u') &= [W_n(t,u) - W_n(t',u)] + [W_n(t',u)-W_n(t',u')] \\ &\geq -[S_n(t,(2u-1)K) - S_n(t',(2u-1)K)] - [S_n(t',(2u-1)K) - S_n(t',(2u'-1)K)] \\ &= -[W_n^o(t,u) - W_n^o(t',u) + (t-t')\ \Sigma_{i=1}^n c_{ni}] - [W_n^o(t',u) - W_n^o(t',u')], \end{aligned}$$

and we also have

$$W_n(t,u) - W_n(t',u') \le W_n(t'',u) - W_n(t',u) - S_n(t,(2u-1)K) + S_n(t'',(2u-1)K)$$

$$+ W_n(t',u'') - W_n(t',u') - S_n(t',(2u-1)K) + S_n(t',(2u''-1)K)$$

$$\le W_n(t'',u'') - W_n(t',u') - S_n(t'',(2u-1)K) + S_n(t'',(2u''-1)K) + S_n(t',(2u-1)K)$$

$$- S_n(t',(2u'-1)K) - S_n(t,(2u-1)K) + S_n(t'',(2u-1)K)$$

$$+ W_n(t',u'') - W_n(t',u') - S_n(t',(2u-1)K) + S_n(t',(2u''-1)K)$$

$$= [W_n(t'',u'') - W_n(t',u')] + [W_n(t',u'') - W_n(t',u')]$$

$$+ [S_n(t'',(2u''-1)K) - S_n(t,(2u-1)K)] + [S_n(t',(2u''-1)K) - S_n(t',(2u'-1)K)]$$

$$= [W_n(t'',u'') - W_n(t',u')] + [W_n(t',u'') - W_n(t',u')]$$

$$+ [W_n^o(t'',u'') - W_n^o(t,u) + (t''-t)\Sigma_{i=1}^n c_{ni})] + [W_n^o(t',u'')-W_n^o(t',u')],$$

Hence,

$$|W_n(t,u) - W_n(t',u')|$$

$$\le 2 \sup\{|W_n(t,u) - W_n(s,v)|;\ (t,u),(s,v) \in L_2(2^m),$$
$$|t-s| \le \delta+2^{-m+1},\ |u-v| \le \delta+2^{-m+1}\}$$

$$+ 2 \sup\{|W_n^o(t,u)-W_n^o(s,v)|;\ (t,u),\ (s,v) \in [0,1]^2,\ |t-s| \le 2^{-m+1},$$
$$|u-v| \le 2^{-m+1}\}$$

$$+ 2^{-m}\ \Sigma_{i=1}^n\ c_{ni}$$

We can treat $[W_n(s'',v'') - W_n(s,v)]$ in the similar way. Therefore, we have

$$\omega_{W_n}(\delta) \le 5 \sup\{|W_n(t,u) - W_n(s,v)|;\ (t,u),\ (s,v) \in L_2(2^m),\ |t-s| \le \delta+2^{-m+1},$$

$$|u-v| \le \delta + 2^{-m+1}\} + 4\,\omega_{W_n^o}(2^{-m+1}) + 2^{-m+1}\,\Sigma_{i=1}^{n} c_{ni}. \qquad (4.7)$$

Since,

$$\omega_{W_n^o}(2^{-m+1}) \le 2\|W_n^o - G\| + \omega_G(2^{-m+1})$$

by Lemma 4.2 and the uniform continuity of G in $[0,1]^2$, we have

$$\omega_{W_n^o}(2^{-m_n+1}) \to 0, \quad \text{as} \quad n \to \infty. \qquad (4.8)$$

where

$$m_n = \max\{m \mid (\max_{1\le i\le n} c_{ni}^{4/3})^{-1} \ge 2^m\} \qquad (4.9)$$

so that $m_n \to \infty$, as $n \to \infty$, and by (4.1),

$$2^{-m_n} \sum_{i=1}^{n} c_{ni} \le M(2^{m_n} \max_{1\le i\le n} c_{ni})^{-1} \le M2^{-m_n} \cdot 2^{3(m_n+1)/4} \to 0, \quad \text{as} \quad n \to \infty$$

So, by virtue of (4.7), it suffices to show that

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} P(\sup\{|W_n(t,u) - W_n(s,v)|;\ (t,u),\ (s,v) \in L_2(2^{m_n}),$$
$$|t-s| \le \delta,\ |u-v| \le \delta\} \ge \epsilon) = 0$$

Let $m(\delta) = \max\{r \mid \delta\, 2^r \ge 1\}$, then

$$\delta\, 2^{m(\delta)} \le 1 \quad \text{and} \quad m(\delta) \to \infty, \quad \text{as} \quad \delta \to 0.$$

By Lemma 4.3, we have

$$\sup\left\{\left| W_n\left[\frac{k_1}{2^{m_n}}, \frac{\ell_1}{2^{m_n}}\right] - W_n\left[\frac{k_2}{2^{m_n}}, \frac{\ell_2}{2^{m_n}}\right]\right|;\right.$$

$$\left. k_i,\ \ell_i = 0,1,\dots,2^{m_n},\quad i=1,2,\ \frac{|k_1-k_2|}{2^{m_n}} \le \delta,\ \frac{|\ell_1-\ell_2|}{2^{m_n}} \le \delta \right\}$$

$$\le \sup\left\{\left|W_n\left(\frac{k_1}{2^{m_n}}, \frac{\ell_1}{2^{m_n}}\right) - W_n\left(\frac{k_2}{2^{m_n}}, \frac{\ell_2}{2^{m_n}}\right)\right|; \right.$$

$$\left. k_i, \ell_i = 0,1,\dots,2^{m_n},\ i=1,2,\ \frac{|k_1-k_2|}{2^{m_n}} \le 2^{-m(\delta)},\ \frac{|\ell_1-\ell_2|}{2^{m_n}} \le 2^{-m(\delta)} \right\}$$

$$\le 8 \sum_{r=m(\delta)}^{m_n} \left[\sup\left\{\left|W_n\left(\frac{k}{2^r}, \frac{\ell}{2^r}\right) - W_n\left(\frac{k+1}{2^r}, \frac{\ell}{2^r}\right)\right|; 0 \le k \le 2^r-1,\ 0 \le \ell \le 2^r \right\} + \right.$$

$$\left. + \sup\left\{\left|W_n\left(\frac{k}{2^r}, \frac{\ell}{2^r}\right) - W_n\left(\frac{k}{2^r}, \frac{\ell+1}{2^r}\right)\right|; 0 \le k \le 2^r,\ 0 \le \ell \le 2^r-1 \right\}\right].$$

Choose $a\in(0,1)$ such that $2^{1/4} a^6 > 1$. Since,

$$\sum_{r=m(\delta)}^{m_n} (1-a)a^{r-m(\delta)} \frac{\epsilon}{16} = \frac{(1-a)\epsilon}{16} \frac{(1-a^{m_n-m(\delta)+1})}{(1-a)} < \frac{\epsilon}{16}.$$

We have

$$P\left[\sup\left\{W_n\left(\frac{k_1}{2^{m_n}}, \frac{\ell_1}{2^{m_n}}\right) - W_n\left(\frac{k_2}{2^{m_n}}, \frac{\ell_2}{2^{m_n}}\right)\right|; \frac{|k_1-k_2|}{2^{m_n}} \le \delta,\ \frac{|\ell_1-\ell_2|}{2^{m_n}} \le \delta \right\} \ge \epsilon \right]$$

$$\le \sum_{r=m(\delta)}^{m_n} \left[P\left[\sup\left\{\left|W_n\left(\frac{k}{2^r}, \frac{\ell}{2^r}\right) - W_n\left(\frac{k+1}{2^r}, \frac{\ell}{2^r}\right)\right|; 0 \le k \le 2^r-1,\ 0 \le \ell \le 2^r \right\}\right.\right.$$

$$\left. > (1-a)a^{r-m(\delta)} \frac{\epsilon}{16} \right]$$

$$+ P\left[\sup\left\{\left|W_n\left(\frac{k}{2^r}, \frac{\ell}{2^r}\right) - W_n\left(\frac{k}{2^r}, \frac{\ell+1}{2^r}\right)\right|; 0 \le k \le 2^r,\ 0 \le \ell \le 2^r - 1 \right\}\right.$$

$$\left.\left. > (1-a)a^{r-m(\delta)} \frac{\epsilon}{16} \right]\right] = \sum_{r=m(\delta)}^{m_n} (I_r + II_r) \tag{4.10}$$

We notice that if a r.v. $\xi$ satisfies

$$E\,\xi^i = \begin{cases} 1 & i = 0 \\ E\,\xi & i \geq 1 \end{cases}.$$

then, for $\ell \geq 2$

$$E(\xi - E\xi)^\ell = \sum_{i=0}^{\ell} \binom{\ell}{i} E\xi^i (E\xi)^{\ell-i} = E\xi \sum_{i=1}^{\ell} \binom{\ell}{i}(E\xi)^{\ell-i} + (E\xi)^\ell$$

$$= E\xi[(1 + E\xi)^\ell - (E\xi)^\ell + (E\xi)^{\ell-1}].$$

Let

$$\Delta_1 = E[W_n(t,u) - W_n(t + 2^{-r},u)]^6 = E[\Sigma_{i=1}^n c_{ni}(\xi_{ni} - E\,\xi_{ni})]^6$$

where $\xi_{ni} = I(F^{-1}(t) + c_{ni}(2u-1)K < Z_i \leq F^{-1}(t + 2^{-r}) + c_{ni}(2u-1)K)$. For $\xi_{ni}$, we have

$$|E(\xi_{ni} - E\,\xi_{ni})^\ell| = \begin{cases} 1 & \ell = 0 \\ 0 & \ell = 1 \\ \leq C\,E\xi_{ni} & 2 \leq \ell \leq 6, \end{cases}$$

where C is a constant. By the choice of $m_n$, we have

$$\max_{1 \leq i \leq n} c_{ni} \leq 2^{-3r/4}, \qquad \text{for } r \leq m_n.$$

So,

$$0 \leq E\xi_{ni} = F(F^{-1}(t+2^{-r}) + c_{ni}(2u-1)K) - F(F^{-1}(t) + c_{ni}(2u-1)K)$$

$$= t + 2^{-r} + F'(\xi)c_{ni}(2u-1)K - t - F'(\eta)\,c_{ni}(2u-1)K$$

$$\leq 2^{-r} + M_1\,c_{ni} \leq M_2\,2^{-3r/4}$$

where $M_1$ and $M_2$ are constants. Hence,

$$\Delta_1 = \sum_{k_1+\ldots+k_n=6} \frac{6!}{k_1!\ldots k_n!} c_{n1}^{k_1} \ldots c_{nn}^{k_n} E(\xi_{n1} - E\xi_{n1})^{k_1} \ldots E(\xi_{nn} - E\xi_{nn})^{k_n}$$

$$= \sum_{i=1}^{n} c_{ni}^6 E(\xi_{ni}-E\xi_{ni})^6 + \frac{6!}{4!2!} \sum_{i\neq j} c_{ni}^4 c_{nj}^2 E(\xi_{ni} - E\xi_{ni})^4 E(\xi_{nj}-E\xi_{nj})^2$$

$$+ \frac{6!}{2!2!2!} \sum_{i\neq j\neq \ell} c_{ni}^2 c_{nj}^2 c_{n\ell}^2 E(\xi_{ni}-E\xi_{ni})^2 E(\xi_{nj}-E\xi_{nj})^2 E(\xi_{n\ell}-E\xi_{n\ell})^2$$

$$+ \frac{6!}{3!3!} \sum_{i\neq j} c_{ni}^3 c_{nj}^3 E(\xi_{ni}-E\xi_{ni})^3 E(\xi_{nj}-E\xi_{nj})^3$$

$$\leq C \sum_{i=1}^{n} c_{ni}^6 E\xi_{ni} + \frac{6!}{4!2!} C^2 \sum_i \sum_j c_{ni}^4 c_{nj}^2 E\xi_{ni} E\xi_{nj}$$

$$+ \frac{6!}{3!3!} C^2 \sum_i \sum_j c_{ni}^3 c_{nj}^3 E\xi_{ni} E\xi_{nj}$$

$$+ \frac{6!}{2!2!2!} C^3 \sum_i \sum_j \sum_\ell c_{ni}^2 c_{nj}^2 c_{n\ell}^2 E\xi_{ni} E\xi_{nj} E\xi_{n\ell}$$

$$\leq C M_2 \max_{1\leq i\leq n} c_{ni}^4 2^{-3r/4} + \frac{6!}{4!2!} C^2 M_2^2 \max_{1\leq i\leq n} c_{ni}^2 2^{-3r/2} + \frac{6!}{2!2!2!} C^3 M_2^3 2^{-9r/4}$$

$$+ \frac{6!}{3!3!} C^2 M_2^2 \max_{1\leq i\leq n} c_{ni}^2 2^{-3r/2} \leq M_3 2^{-9r/4},$$

where $M_3$ is a constant. Therefore,

$$I_r = P\left[ \sup\left|W_n\left( \frac{k}{2^r}, \frac{\ell}{2^r} \right) - W_n\left( \frac{k+1}{2^r}, \frac{\ell}{2^r} \right)\right| > (1-a)a^{r-m(\delta)} \frac{\epsilon}{16} \right]$$

$$\leq \sum_{\ell=0}^{2^r} \sum_{k=0}^{2^r-1} P\left[\left|W_n\left( \frac{k}{2^r}, \frac{\ell}{2^r} \right) - W_n\left( \frac{k+1}{2^r}, \frac{\ell}{2^r} \right)\right| > (1-a)a^{r-m(\delta)} \frac{\epsilon}{16} \right]$$

$$\leq \sum_{\ell=0}^{2^r} \sum_{k=0}^{2^r-1} \left[ \frac{16}{\epsilon(1-a)a^{r-m(\delta)}} \right]^6 E\left[W_n\left( \frac{k}{2^r}, \frac{\ell}{2^r} \right) - W_n\left( \frac{k+1}{2^r}, \frac{\ell}{2^r} \right)\right]^6$$

$$\le \sum_{\ell=0}^{2^r} \sum_{k=0}^{2^r-1} \left[ \frac{16}{\epsilon(1-a)a^{r-m(\delta)}} \right]^6 M_3 \, 2^{-9r/4} \le 2 \, M_3 \left[ \frac{16}{\epsilon(1-a)a^{r-m(\delta)}} \right]^6 2^{-r/4} \tag{4.11}$$

Similarly, let

$$\Delta_2 = E[W_n(t,u) - W_n(t,u+2^{-r})]^6 = E\left[ \sum_{i=1}^{n} c_{ni}(\eta_{ni} - E\eta_{ni}) \right]^6$$

where

$$\eta_{ni} = I(F^{-1}(t) + c_{ni}(2u-1)K < Z_i \le F^{-1}(t) + c_{ni}(2u+2^{-r+1}-1)K).$$

Then,

$$E\eta_{ni} = F(F^{-1}(t) + c_{ni}(2u+2^{-r+1}-1)K) - F(F^{-1}(t) + c_{ni}(2u-1)K)$$

$$= F'(\xi) \, K \, c_{ni} \, 2^{-r+1} \le M_4 \, 2^{-3r/4}$$

where $M_4$ is a constant. Therefore,

$$II_r \le 2 \, M_5 \left[ \frac{16}{\epsilon(1-a)a^{r-m(\delta)}} \right]^6 2^{-r/4} \tag{4.12}$$

where $M_5$ is a constant.

From (4.10), (4.11) and (4.12), we have

$$P\left[ \sup\left\{ \left| W_n\left( \frac{k_1}{2^{m_n}}, \frac{\ell_1}{2^{m_n}} \right) - W_n\left( \frac{k_2}{2^{m_n}}, \frac{\ell_2}{2^{m_n}} \right) \right| ; \frac{|k_1-k_2|}{2^{m_n}} \le \delta, \frac{|\ell_1-\ell_2|}{2^{m_n}} \le \delta \right\} \ge \epsilon \right]$$

$$\le 2(M_3+M_5) \sum_{r=m(\delta)}^{m_n} \left[ \frac{16}{\epsilon(1-a)a^{r-m(\delta)}} \right]^6 2^{-r/4}$$

$$\le \frac{2(M_3+M_5)}{(2^{1/4})^{m(\delta)}} \left[ \frac{16}{\epsilon(1-a)} \right]^6 \sum_{k=0}^{\infty} \rho^k \to 0, \quad \text{as } \delta \to 0.$$

because $\rho = (a^6\, 2^{1/4})^{-1} < 1$ and $m(\delta) \to \infty$, as $\delta \to 0$. □

PROPOSITION 4.5. Let $T_n(t,u) = \bar{S}_n^*(t,(2u-1)K) - t \sum_{i=1}^{n} c_{ni}$, and let $\{P_n;\ n \geq 1\}$ be the sequence of probability measures corresponding to $T_n$, $n \geq 1$. Then, $\{P_n\}$ is relatively compact.

Proof. Note that $T_n(t,u) \in C[0,1]^2$, $n \geq 1$. Hence, it suffices to show that for any $\epsilon > 0$,

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} P(\omega_{T_n}(\delta) \geq \epsilon) = 0 \qquad (4.13)$$

Since,

$$|T_n(t,u) - T_n(s,v)| = |[\bar{S}_n^*(t,(2u-1)K) - \bar{S}_n^*(s,(2v-1)K)] - (t-s) \sum_{i=1}^{n} c_{ni}|$$

$$\leq |[\bar{S}_n^*(t,(2u-1)K) - \bar{S}_n^*(s,(2v-1)K)] - [S_n^*(t,(2u-1)K) - S_n^*(s,(2v-1)K)]|$$

$$+ |W_n(t,u) - W_n(s,v)| + |W_n^o(t,u) - W_n^o(s,v)|$$

by (4.5), we have

$$\omega_{T_n}(\delta) \leq 4 \max_{1\leq i\leq n} c_{ni} + \omega_{W_n}(\delta) + \omega_{W_n^o}(\delta), \qquad \text{a.s.}$$

Therefore, (4.13) follows from (4.2), (4.8) and Proposition 4.4.

Since $S_n^*(0,-K) \equiv 0$, by (4.5), we have

$$|T_n(0,0)| = |\bar{S}_n^*(0,-K)| \leq 2 \max_{1\leq i\leq n} c_{ni}, \qquad \text{a.s.}$$

Hence,

$$P_n^o = P\, \pi_o^{-1} \qquad \text{converges in distribution.} \qquad (4.14)$$

By virtue of Neuhaus [5] (discussion on pp. 1290-1291), (4.13) and (4.14) imply that $\{P_n\}$ is relatively compact. □

LEMMA 4.6. Suppose $\Gamma = \{T_\lambda : \lambda \in \Lambda\}$ is a compact set in $C[0,1]^2$, let $\Gamma_1 = \{T_\lambda(\cdot,u) : \lambda \in \Lambda,\ u \in [0,1]\}$, then $\Gamma_1$ is a compact set in $C[0,1]$.

Proof. It suffices to show that any infinite sequence in $\Gamma_1$ contains a convergent subsequence in $\Gamma_1$.

Suppose $\{T_{\lambda_n}(\cdot,u_n)\}$ is a sequence of $\Gamma_1$, then $\{T_{\lambda_n}(\cdot,\cdot)\}$ is a sequence of $\Gamma$. Since $\Gamma$ is compact, $\{T_{\lambda_n}(\cdot,\cdot)\}$ contains a convergent subsequence also denoted by $\{T_{\lambda_n}(\cdot,\cdot)\}$ such that

$$\|T_{\lambda_n} - T_{\lambda_o}\| \to 0, \qquad \text{as } n \to \infty \tag{4.15}$$

where $\lambda_o \in \Lambda$. Since $u_n \in [0,1]$, there exists a convergent subsequence also denoted by $u_n$ such that

$$u_n \to u_o \in [0,1], \qquad \text{as } n \to \infty. \tag{4.16}$$

For any $t \in [0,1]$, we have

$$|T_{\lambda_n}(t,u_n) - T_{\lambda_o}(t,u_o)| \leq |T_{\lambda_n}(t,u_n) - T_{\lambda_o}(t,u_n)| + |T_{\lambda_o}(t,u_n) - T_{\lambda_o}(t,u_o)|$$

Therefore, by (4.15), (4.16) and the uniform continuity of $T_{\lambda_o}$ in $[0,1]^2$, we have

$$\|T_{\lambda_n}(\cdot,u_n) - T_{\lambda_o}(\cdot,u_o)\| \to 0, \qquad \text{as } n \to \infty$$

where $T_{\lambda_o}(\cdot,u_o) \in \Gamma_1$. □

PROPOSITION 4.7. If for any compact set $\Gamma_o$ in $C[0,1]$

$$\lim_{t\to 0} \frac{\mathrm{Rem}(tH)}{t} = 0 \tag{4.17}$$

uniformly for $H \in \Gamma_o$, then for $K > 0$,

$$\sup_{|u|\le K} \left| \sum_{i=1}^{n} c_{ni} \, \mathrm{Rem}\left[ \frac{\vec{S}_n^*(\cdot,u)}{\Sigma_{i=1}^n c_{ni}} - U(\cdot) \right] \right| \xrightarrow{P} 0, \quad \text{as} \quad n \to \infty. \tag{4.18}$$

Proof. From Proposition 4.5, we know that $\{P_n\}$ is relatively compact in $C[0,1]^2$, where

$$P_n(A) = P(T_n \in A).$$

Since $C[0,1]^2$ is complete and seperable, by Prohorov's theorem (Billingsley [1], Theorem 6.2), $\{P_n\}$ is tight, i.e., for any $\epsilon > 0$, there exists a compact set $\Gamma$ in $C[0,1]^2$ such that

$$P(T_n \in \Gamma) > 1-\epsilon, \qquad \text{for } n \ge 1. \tag{4.19}$$

Use the same definition about $\Gamma_1$ in Lemma 4.6 with respect to $\Gamma$, we know that $\Gamma_1$ is a compact set of $C[0,1]$. By (4.3) and (4.17), there exists N such that

$$\left| \sum_{i=1}^{n} c_{ni} \, \mathrm{Rem}\left[ \frac{H}{\Sigma_{i=1}^n c_{ni}} \right] \right| \le \epsilon, \quad \text{for } n \ge N, \;\; H \in \Gamma_1 \tag{4.20}$$

If $T_n \in \Gamma$, then, $T_n(\cdot,u) \in \Gamma$, for any $u \in [0,1]$ and by (4.20),

$$\left| \sum_{i=1}^{n} c_{ni} \, \mathrm{Rem}\left[ \frac{T_n(\cdot,u)}{\Sigma_{i=1}^n c_{ni}} \right] \right| \le \epsilon, \quad \text{for } n \ge N, \quad u \in [0,1]. \tag{4.21}$$

(4.21) implies

$$\sup_{0\le u\le 1} \left| \sum_{i=1}^{n} c_{ni} \, \mathrm{Rem}\left[ \frac{T_n(\cdot,u)}{\Sigma_{i=1}^n c_{ni}} \right] \right| \le \epsilon, \quad \text{for } n \ge N. \tag{4.22}$$

Equivalently, (4.22) is written as

$$\sup_{|u|\le K} \left| \sum_{i=1}^{n} c_{ni} \operatorname{Rem}\left[ \frac{\bar{S}_n^*(\cdot,u)}{\Sigma_{i=1}^n c_{ni}} - U(\cdot)\right]\right| \le \epsilon, \qquad \text{for } n \ge N. \tag{4.23}$$

Therefore, from (4.19) and (4.23), we have, for $n \ge N$,

$$1 - \epsilon < P(T_n \in \Gamma) \le P\left[ \sup_{|u|\le K} \left| \sum_{i=1}^{n} c_{ni} \operatorname{Rem}\left[ \frac{\bar{S}_n^*(\cdot,u)}{\Sigma_{i=1}^n c_{ni}} - U(\cdot) \right]\right| \le \epsilon \right]$$

□

Let $\Gamma$ be a set in $D[0,1]$ and $H \in D[0,1]$, define

$$\operatorname{dist}(H,\Gamma) = \inf_{G\in\Gamma} \|H-G\| \tag{4.24}$$

LEMMA 4.8. Let $Q : D[0,1] \times R \to R$ and suppose that for any compact set $\Gamma$ in $D[0,1]$

$$\lim_{t\to 0} Q(H,t) = 0 \tag{4.25}$$

uniformly for $H \in \Gamma$. Let $\epsilon > 0$ and let $\alpha_n$, $\beta_n$ be sequences of real numbers such that $\alpha_n \to 0$, $\beta_n \to 0$, as $n \to \infty$. Then, for any compact set $\Gamma$ in $D[0,1]$, there exists $N$ such that, for $n \ge N$, if $\operatorname{dist}(H,\Gamma) \le \alpha_n$, then

$$|Q(H,\beta_n)| < \epsilon.$$

Proof. Suppose not. Then, for $\epsilon > 0$, there exists a compact set $\Gamma$ in $D[0,1]$ and sequence $\{H_k\} \subset D[0,1]$ with $\operatorname{dist}(H_k,\Gamma) \le \alpha_{n_k}$ such that

$$|Q(H_k, \beta_{n_k})| \ge \epsilon. \tag{4.26}$$

Since $\operatorname{dist}(H_k,\Gamma) \le \alpha_{n_k}$, we can choose $H_k^* \in \Gamma$ such that

$$\|H_k - H_k^*\| \le \alpha_{n_k}.$$

Since $\{H_k^*\} \subset \Gamma$ and $\Gamma$ is a compact set, then $\{H_k^*\}$ has an accumulation point $H^* \in \Gamma$. Therefore, we can choose a subsequence of $\{H_k^*\}$ also denoted by $\{H_k^*\}$ such that

$$H_k^* \to H^* \quad \text{as} \quad k \to \infty.$$

Since $\alpha_{n_k} \to 0$, we also have

$$H_k \to H^* \qquad \text{as } k \to \infty$$

and the set

$$\Gamma_1 = \{H_k : k \geq 1\} \cup \{H^*\}$$

is compact. By (4.25), we have

$$Q(H_k, t) \to 0, \quad \text{as } t \to 0.$$

uniformly for $H_k \in \Gamma$. This contradicts (4.26). □

From the discussion by Fernholz [2], we know that $[S_n^*(\cdot,u) - U(\cdot)\sum_{i=1}^{n} c_{ni}]$ are not random elements of D[0,1] with uniform topology. Next, we will use the inner probability measure $P_*$ corresponding to P to deal with this problem.

LEMMA 4.9. Under the assumption of Proposition 4.7, for $\epsilon > 0$, there exists a compact set $\Gamma$ in D[0,1] such that for all $n \geq 1$

$$P_*(\text{dist}([S_n^*(\cdot,(2u-1)K) - U(\cdot)\Sigma_{i=1}^n c_{ni}], \Gamma) \leq 2 \max_{1 \leq i \leq n} c_{ni}, \ \forall\, u \in [0,1]) > 1-\epsilon.$$

Proof. As the proof of Proposition 4.7, there exists a compact set $\Gamma$ in $C[0,1]^2$ such that for all $n \geq 1$,

$$P(T_n \in \Gamma) > 1 - \epsilon.$$

By (4.5), we have, for $n \geq 1$,

$$P(T_n \in \Gamma, \|\bar{S}_n^*(\cdot,\cdot) - S_n^*(\cdot,\cdot)\| \leq 2 \max_{1 \leq i \leq n} c_{ni}) \geq 1 - \epsilon. \qquad (4.27)$$

By Lemma 4.6, we know that $\Gamma$ induces a compact set $\Gamma_1$ in C[0,1], which is also a compact set of D[0,1] because C[0,1] is a subspace of D[0,1]. We also know that if $T_n \in \Gamma$, then $T_n(\cdot,u) \in \Gamma_1$ for any $u \in [0,1]$, i.e.,

$$[\bar{S}_n^*(\cdot,(2u-1)K) - U(\cdot)\Sigma_{i=1}^n c_{ni}] \in \Gamma_1, \text{ for any } u \in [0,1]. \qquad (4.28)$$

Also, $\|\bar{S}_n^*(\cdot,\cdot) - S_n^*(\cdot,\cdot)\| \leq 2 \max_{1 \leq i \leq n} c_{ni}$ implies

$$\|\bar{S}_n^*(\cdot,(2u-1)K) - S_n^*(\cdot,(2u-1)K\| \leq 2 \max_{1 \leq i \leq n} c_{ni}, \ \forall\, u \in [0,1]. \qquad (4.29)$$

Since (4.28) and (4.29) imply

$$\text{dist}([S_n^*(\cdot,(2u-1)K) - U(\cdot)\Sigma_{i=1}^n c_{ni}], \Gamma_1) \leq 2 \max_{1 \leq i \leq n} c_{ni}, \quad \forall\, u \in [0,1],$$

by (4.27), we have, for $n \geq 1$,

$$P_*(\text{dist}([S_n^*(\cdot,(2u-1)K)-U(\cdot)\Sigma_{i=1}^n c_{ni}], \Gamma_1) \leq 2 \max_{1 \leq i \leq n} c_{ni}, \ \forall\, u \in [0,1]) > 1-\epsilon.$$

□

<u>Proof of Theorem 3.1.</u> Since $\tau : D[0,1] \to R$ is Hadamard differentiable at U, (4.17) holds.

By Lemma 4.9, for $\epsilon > 0$, there exists a compact set $\Gamma$ in D[0,1] such that, for $n \geq 1$

$$P_*(\text{dist}([S_n^*(\cdot,(2u-1)K)-U(\cdot)\Sigma_{i=1}^n c_{ni}], \Gamma) \leq 2 \max_{1 \leq i \leq n} c_{ni}, \ \forall\, u \in [0,1]) > 1-\epsilon/2.$$

Therefore, we can find measurable sets $E_n$ for all n such that

$$E_n \subset \{\text{dist}([S_n^*(\cdot,(2u-1)K) - U(\cdot)\Sigma_{i=1}^n c_{ni}],\Gamma) \le 2 \max_{1\le i\le n} c_{ni},\ \forall\, u \in [0,1]\}$$

and

$$P(E_n) > 1 - \epsilon. \tag{4.30}$$

By Lemma 4.8, consider $Q(H,t) = \frac{\text{Rem}(tH)}{t}$. For the compact set $\Gamma$, there exists N such that, for $n \ge N$, if $\text{dist}(H,\Gamma) \le 2 \max_{1\le i\le n} c_{ni}$, then

$$\left|\sum_{i=1}^n c_{ni}\ \text{Ren}\left[\frac{H}{\Sigma_{i=1}^n c_{ni}}\right]\right| < \epsilon.$$

Therefore, for $n \ge N$, take $H = S_n^*(\cdot,(2u-1)K) - U(\cdot)\Sigma_{i=1}^n c_{ni}$ for any $u \in [0,1]$, $\{\text{dist}([S_n^*(\cdot,(2u-1)K) - U(\cdot)\Sigma_{i=1}^n c_{ni}],\ \Gamma) \le 2 \max_{1\le i\le n} c_{ni}$, $\forall\, u \in [0,1]\}$ implies

$$\left|\sum_{i=1}^n c_{ni}\ \text{Rem}\left[\frac{S_n^*(\cdot,(2u-1)K)}{\Sigma_{i=1}^n c_{ni}} - U(\cdot)\right]\right| < \epsilon, \quad \text{for } u \in [0,1]. \tag{4.31}$$

Since (4.31) implies

$$\sup_{|u|\le K}\left|\sum_{i=1}^n c_{ni}\ \text{Rem}\left[\frac{S_n^*(\cdot,u)}{\Sigma_{i=1}^n c_{ni}} - U(\cdot)\right]\right| \le \epsilon,$$

by (4.30), we have, for $n \ge N$,

$$1 - \epsilon < P(E_n) \le P\left[\sup_{|u|\le K}\left|\Sigma_{i=1}^n c_{ni}\ \text{Rem}\left[\frac{S_n^*(\cdot,u)}{\Sigma_{i=1}^n c_{ni}} - U(\cdot)\right]\right| \le \epsilon\right].$$

□

Remark (1). In the proof of Theorem 3.1, we assumed that $\text{Rem}\left[\frac{S_n^*(\cdot,u)}{\Sigma_{i=1}^n c_{ni}} - U(\cdot)\right]$ is measurable, even though $S_n^*(\cdot,u)$ is not a random element of D[0,1]. This is because that $\tau\left[\frac{S_n^*(\cdot,u)}{\Sigma_{i=1}^n c_{ni}}\right]$ is measurable, and by Lemma 4.4.1 of Fernholz [2], we know that $\tau_U'(S_n^*(\cdot,u) - U(\cdot)\Sigma_{i=1}^n c_{ni})$ is also measurable, hence $\text{Rem}\left[\frac{S_n^*(\cdot,u)}{\Sigma_{i=1}^n c_{ni}} - U(\cdot)\right]$ is measurable.

Remark (2). By Lemma 4.4.1 of Fernholz [2], we have

$$\tau_U'(S_n^*(\cdot,u) - U(\cdot)\Sigma_{i=1}^n c_{ni}) = \Sigma_{i=1}^n c_{ni}\ \tau_U'((\delta_{(Z_i-c_{ni}u)} - F) \circ F^{-1})$$

$$= \sum_{1=1}^{n} c_{ni}\ IC(Z_i-c_{ni}u;\ F,\ T).$$

Note that this is true without any assumptions on $\{c_{ni}\}$ and F as long as $\tau$ is Hadamard differentiable at U.

Remark (3). If in Theorem 3.1 and Proposition 4.7,

$$\sup_{|u|\le K} \left|\Sigma_{i=1}^n c_{ni}\ \text{Rem}\left[\frac{S_n^*(\cdot,u)}{\Sigma_{i=1}^n c_{ni}} - U(\cdot)\right]\right|$$

or

$$\sup_{|u|\le K} \left|\Sigma_{i=1}^n c_{ni}\ \text{Rem}\left[\frac{\bar{S}_n^*(\cdot,u)}{\Sigma_{i=1}^n c_{ni}} - U(\cdot)\right]\right|$$

is not measurable, we replace $|u| \le K$ by $u \in Q_K$, where $Q_K$ = {all rational numbers in [-K,K]}, then, they both are measurable. The results will be slightly different, but still good enough for the study of (1.15).

## References


[1] Billingsley, P. (1968). *Convergence of Probability Measures*. John Wiley & Sons, New York.

[2] Fernholz, L.T. (1983) *Von Mises Calculus for Statistical Functional*. Lecture Notes in Statistics 19. Springer, New York. [1.3f, 2.1b, 2.3a]

[3] Jurecková, J. (1984) M-, L- and R-estimators. In Handbook of Statistics, Volume 4: Nonparametric Methods (eds.: P.R. Krishnaiah and P.K. Sen), North Holland, Amsterdam pp. 463-485.

[4] Kallianpur, G. (1963) Von Mises function and maximum likelihood estimation. *Sankhyā* A 23, 149-158 [2.3a]

[5] Neuhaus, G. (1971) On weak convergence of stochastic processes with multidimensional time parameter. *AMS* 42, 1285-1295.

[6] Reeds, J.A. (1976) On the definition of Von Mises functionals. Ph.D. dissertation. Harvard Univ., Cambridge, MA.

[7] Von Mises, R. (1947) On the asymptotic distribution of differentiable statistical functions. *AMS* 18, 309-348.