

AN APPLICATION OF MATRIX ALGEBRA TO THE STUDY
OF HUMAN RELATIONS WITHIN ORGANIZATIONS

by

Leo Katz, Ph. D.

Special report to the Office of Naval
Research of work at Chapel Hill under
Project NR 042 031

Institute of Statistics
Mimeograph Series 38
For Limited Distribution

Final report on work on mathematical analysis of social group organization under ONR contract at Department of Mathematical Statistics, University of North Carolina, Chapel Hill. June 15 - August 15, 1950.

The Problem. Group organization evolving from interpersonal contacts has been studied systematically for some twenty years by a group of sociometrists, most of whose publications appear in the journal, Sociometry. J. L. Moreno invented as a tool for this study the sociometric test, which is still the basic instrument. The test is applied to a well-defined, organic, social group as follows: Each of the N individuals in the group is asked to name a number (specified or unspecified) of the others with whom he would prefer to be associated in a particular activity. The motivation for careful choice should be very strong and secrecy should be guaranteed. In addition, each person may be asked to name those with whom he does not wish to be associated and/or may be asked to make selections separately for more than one activity. The result is a complete listing of the responses of each person to each other in the group, counting "indifference" as a response.

The data collected may be exhibited in two forms. The first, the sociogram, has been explored extensively in Sociometry and is essentially descriptive in character. Since it is not analytic, it will not be mentioned further in this report. The second form is that of a matrix of choices, C , in which the element c_{ij} is the numerical representation of the response of the i -th individual to the j -th. The principal diagonal elements, c_{ii} , are usually taken to be zero.

There are two broad classes of problems, both of which are largely unsolved in any mathematical or statistical sense. In problems of the first kind, one tries to construct an index which is a function of the elements in the matrix and expresses in summary fashion some characteristic of the whole group. Construction of each such measure immediately introduces the secondary

problem of the distribution of the index under some suitable null hypothesis. Problems of the second kind deal with the internal structural organization of the group. In these problems, one asks how the group structures itself as opposed to how it is externally structured by such things as physical, socio-economic and ethnic differences.

The measurement of cleavage illustrates the difference between the two kinds of problems. We may have subgroups defined externally, e. g., by sex or as owner or tenant, etc. Cleavage is then measured by the relative intensities of inter-group and intra-group choices. This is a problem of the first kind. On the other hand, starting with a group which is fairly homogeneous by external considerations, we often observe that subgroups or cliques form as a result of interpersonal relations. In this case, we may use the concept of cleavage inversely to delineate the cliques and define the internal structure. We would say that the internal structure is that partitioning of individuals into subgroups which maximizes the cleavage (in an appropriate sense).

Applications. There are many applications of the sociometric technique to industrial and military operations as well as to sociological research. A few of these are mentioned below in some detail.

1. Leaders. It is important that rational methods be developed to select those individuals who, while functionally part of the group, have assumed or had thrust upon them the roles of informal leaders. This problem is particularly acute in the selection of "straw-bosses" or "non-coms." The methods should be relatively simple, certainly not of a sort requiring extensive case study of each group. The usual approach to this problem is to consider the number of choices received by each person, $s_j = \sum_{i=1}^N c_{ij}$. The difficulty with that approach is that a popularity contest may not necessarily

3 - An application of matrix algebra to the study of human relations within - 3
organizations - Katz

reveal the true leader in a more complex situation in which he exercises some control over a few key individuals who in turn control the rest of the group. The structural analysis necessary for such cases would work uniformly for all cases.

2. Selection of teams. Quite often, at the end of a period of training or indoctrination, it is desirable to break a group down into smaller units which are to function in a manner in which it is important that there be much mutual good-will and understanding and little friction within the units. This is part of a larger problem discussed later.

3. Chains of communication of information, rumors, etc. In any analysis in which it is necessary to consider the chains formed by combining elementary one-directional links between individuals, the appropriate tools are the powers of the choice matrix. For example, the typical element of

c^2 is $c_{ij}^{(2)} = \sum_k c_{ik} c_{kj}$, the number of chains of length two from i to j .

The higher powers, similarly, give the chains of greater lengths. We may then speak of horizontal and of vertical structures according as there are many or few chains which are closed (e. g., $i \rightarrow j \rightarrow k \rightarrow i$) as compared with those of equal length which are not closed. Further, we may discuss decomposition or separability of the group in terms of the chains.

4. Isolates. The true isolate in a social group is always a potential source of trouble. Since he is not accepted by any of the other members of the group, it becomes almost impossible to make a reasonably effective partitioning of the group into smaller units. However, in the usual forms of the sociometric test, it is possible to observe a small number of apparent isolates by chance. It is, therefore, necessary that the distribution of the number of isolates under the null hypothesis be known in order to judge whether the observed

number is excessive.

5. Types of structure. Construction of a typology of group organizations is a task perhaps best left to the social scientists, but there are other problems in this general area. One problem arises in evaluation of a training program or educational process designed to produce a specified kind of group. Examples of such programs are education for democratic living and military training to produce smoothly functioning combat teams such as tank or aircraft crews. Programs of this sort have seldom or never been evaluated directly in terms of the groups they actually produce. Another problem is that of construction of a few indexes of type of group organization together with derivation of the appropriate sampling distributions. Such indexes would be a helpful preliminary to the solution of the typology problem. An example of this kind of index is one which would measure democratic versus authoritarian group structure.

Results and formulations of problems for further study. The first problem attacked was that of the chance distribution of the number of isolates, those who were not chosen by any of the others in the group. This problem was treated combinatorially and an exact solution obtained for the case of equal likelihood of choice. The exact distribution being rather involved, a simpler approximation was sought and was obtained. It turned out that a particular binomial approximation was useful even for quite small groups. For complete details of this part of the work, see Institute of Statistics Mineo Series 36, entitled "The distribution of the number of isolates in a group." This paper was also submitted to the Annals of Mathematical Statistics for possible publication.

The next problem taken up was that of the distribution of the number

of mutual or reciprocated choices, e.g., A chooses B and B chooses A. If we call A and B a mutual pair, there are two reciprocated choices in the pair. It was considered best to discuss r , the number of reciprocated choices. $r = 2$ (mutual choices). In the matrix C^2 , the element $c_{ii}^{(2)}$ = the number of reciprocated choices made by the i -th individual. Thus, $r = \text{tr}(C^2)$. Several attempts were made to obtain the distribution of r directly, using the notion of a matrix C with random elements. The difficulty here seems to be that, while the elements of C are random, they are not independent. Another attempt to solve this problem took the following form. For each individual, collate the appropriate row and column of C . Then the number of his choices which are reciprocated is equal to the number of matched choices in the collation. Assume (i) the d outgoing choices (row) are uniformly distributed so the probability of observing a choice in any one cell is $\frac{d}{N-1}$, (ii) the s_i incoming choices (column) are uniformly distributed independently of the outgoing choices, and (iii) the number of reciprocated choices for the i -th individual is independent of the number for the j -th. Assumptions (i) and (ii) are realistic; the third is not. Using the third, along with the other two, it is possible (by brute force) to obtain a few moments of the distribution of r and to show that the asymptotic distribution is very likely Poisson. However, it is also possible to show that, for moderate values of N , the distribution may be very far from the asymptotic form and depends heavily upon the distribution of the s_i ; $i = 1, 2, \dots, N$.

The results obtained may be useful for large groups, in which the third assumption is very nearly true, but are practically useless for smallish groups. Further research in process is directed at weakening the troublesome third assumption. Thus, the results on this problem are incomplete.

In studying the chain structure of the group, the totality of chains of

length k is given by C^k . If C^k is post-multiplied by C , we obtain the chains of length $k + 1$ by the process of adding a link to the head end of a k -chain to form a $(k + 1)$ -chain. The new chain is said to be "redundant" if the new head individual coincides with one of the preceding individuals in the chain. Luce and Perry (Psychometrika, vol. 14, 1949) discussed the problem and showed how to obtain non-redundant chains of length 3; non-redundant chains of length 4 may be obtained as follows: Let $P^{(k)}$ be the matrix of the non-redundant chains of length k , so that $p_{ij}^{(k)}$ is the number of non-redundant k -chains from i to j . $P^{(3)}$ is obtained by the method of Luce and Perry. Then

$$CP^{(3)} = F_1^{(4)} + F_2^{(4)} + F_3^{(4)} + F_4^{(4)}$$

where each chain of $P^{(3)}$ is a 3-chain connecting α_1 to α_2 to α_3 to α_4 and each chain of C is a 1-chain α_0 to α_1 . We have resulting chains of four different types represented by the F 's above. They are

1. $\alpha_0 \neq \alpha_1, \quad i = 2, 3, 4 \quad (F_1^{(4)})$
2. $\alpha_0 = \alpha_2 \quad (F_2^{(4)})$
3. $\alpha_0 = \alpha_3 \quad (F_3^{(4)})$
4. $\alpha_0 = \alpha_4 \quad (F_4^{(4)})$.

Thus, $F_1^{(4)} = P^{(4)}$, the required matrix of non-redundant 4-chains. This is obtained by subtraction of all of the remaining F 's from the left hand member. $F_4^{(4)}$ is the principal diagonal of $CP^{(3)}$ and is obtained trivially. The other two are

$$F_2^{(4)} = F_2^{(2)} P^{(2)} - AC - B$$

where $a_{ij} = c_{ij}c_{ji}$ and $b_{ij} = c_{ij}c_{ji}c_{ij}^{(2)}$ and

$$F_3^{(4)} = F_3^{(3)}C - D$$

where $d_{ij} = c_{ij} [c_{ji}^{(2)} + c_{ij}^{(2)} c_{ji}]$.

Thus far, attempts to write a completely general form for $P^{(k)}$ have been fruitless, although it is to be expected that a similar approach to the one used in obtaining the above result may give the desired solution. Certainly the iterative formulation of the problem is more compact than that of Luce and Perry.

The first approach to the problems of internal structure was through the work of Frobenius [Sitzungsberichte preuss. Ak. Wiss., 1912] on matrices of non-negative elements. Frobenius defined decomposition of a matrix as follows: A matrix, M , is decomposable if, through simultaneous permutations of the rows and the corresponding columns (order permutations), it may be exhibited in the form

$$M = \begin{bmatrix} P & U \\ V & Q \end{bmatrix},$$

where either $U = 0$ or $V = 0$. If both are zero matrices, M is completely decomposable. If a matrix is decomposable, we may speak of decomposition into m indecomposable parts, if

$$M = \begin{bmatrix} M_{11} & M_{12} & \dots & M_{1m} \\ M_{21} & M_{22} & \dots & M_{2m} \\ \vdots & & \ddots & \\ M_{r1} & \dots & \dots & M_{rm} \end{bmatrix}$$

and each M_{ii} is indecomposable while each M_{ij} with $i > j$ is a zero matrix.

The application of the notion of matrix decomposition to group structure

is obvious. Theoretically, if a group has a vertical form of organization, we should expect choices to be relatively dense within strata corresponding to the indecomposable principal minors with fewer choices directed to "higher" strata. Since no choices (theoretically) go to lower strata, all of the blocks M_{ij} , $i > j$, are empty. In practice, however, we will have blocks which are approximately empty below the principal minor blocks and we should need a fairly complex sampling theory to account for these. The worst features of this problem are that we should then require a new set of definitions of almost decomposability and that Frobenius' results would not apply.

Another approach to the problem of structure and also to the problem of breaking a group into teams in some "best" manner is the following. Consider an $N \times n$ grouping matrix G , such that $g_{ij} = 0$ for every $j = 1, 2, \dots, m$ except one, say j' . Individual i then is said to belong to subgroup j' , and $g_{ij'} \neq 0$ for each i corresponding to a member of the j' subgroup. Hotelling, in a report to the Chief of Naval Research, dated October 1, 1948, pointed out that the trace of $G'CG$ measures the aggregate good fortune of the group when assignments are made into m equal subgroups. The good fortune of an individual is defined to be the number of others actually in his subgroup whom he wanted to be in his subgroup, and the non-zero g_{ij} are each assigned unit value. The problem of assignment becomes one of maximizing the trace through proper selection of G .

When the number in each group is not fixed in advance, i.e., when we seek to discover the internal structure, the problem is somewhat more complicated. If the g_{ij} which are non-zero are taken equal to unity, the maximum trace corresponds to $g_{11} \cong 1$, i.e., all individuals in one group. The best way out of this difficulty appears to be to take

9 - An application of matrix algebra to the study of human relations within - 9
 organizations - Katz

$$g_{ij} = \frac{1}{n_j}, \text{ if } i \text{ is in the } j\text{-th subgroup and}$$

$$= 0, \text{ otherwise.}$$

The bad feature of this choice is that the optimum good fortune of any individual is $\frac{n_j-1}{n_j}$ and depends on the size of the group in which he finds himself.

However, if we choose the non-zero $g_{ij} = \frac{1}{n_j-1}$ so that the optimum individual good fortune is 1, we get into serious difficulties with subgroups of one (isolates). Also, it turns out that the choice of $\frac{1}{n_j}$ gives very simple latent roots for one of the critical matrices.

The latent roots of $G'CG$ are the same as the latent roots of $GG'C$. Therefore, we consider $H = GG'$, an $N \times N$ matrix. We write $H = PBP$, where P is a permutation matrix and

$$B = \begin{bmatrix} \frac{1}{n_1} & \frac{1}{n_1} & \dots & \frac{1}{n_1} \\ \vdots & & & \vdots \\ \frac{1}{n_1} & \dots & & \frac{1}{n_1} \\ & & & \\ & & \frac{1}{n_2} & \dots & \frac{1}{n_2} \\ & & \vdots & & \vdots \\ & & \frac{1}{n_2} & \dots & \frac{1}{n_2} \\ & & & & \end{bmatrix},$$

consisting of m principal diagonal blocks of order n_i with elements $1/n_i$ such that $\sum_1^m n_i = N$. The latent roots of B (and hence, of H) are m 1's and $(N-m)$ 0's.

Frobenius (Sitzungsberichte preuss. Ak. Wiss., 1896) showed that, for commutative matrices, any rational function of the matrices has characteristic roots which are the same function of sets of the characteristic roots of the

10-An application of matrix algebra to the study of human relations within-10
organizations - Katz

individual matrices and that the ordering which produces the sets is the same for every rational function. If this theorem or a similar one held for the product HC , our problem would be solved. That the theorem as stated does not hold can be immediately shown by construction of counterexamples. H and C are not commutative; the question we should like answered is: Since we know the latent roots of H and may observe C and obtain its latent roots, can we express the latent roots and hence the trace of HC in terms of the roots of H and of C ?

This, then, is an unsolved problem of great importance in this theory. H is a symmetric matrix and, to within an order permutation, of the form B . C , on the other hand, is not even symmetric. However, since H is symmetric, the latent roots of HC' are the same as those of HC and the trace of HD , where $D = C + C'$ is twice that of HC , so that the problem may be stated in terms of two symmetric matrices. Thus we may restate the purely algebraic problem in two parts as follows.

1. What is the relationship of the characteristic roots of HC to h_1, h_2, \dots, h_n and $c_{\alpha_1}, c_{\alpha_2}, \dots, c_{\alpha_n}$, the roots of H and of C , respectively, if H is of the form of B to within order permutations?

2. If the relationship is of the form $\lambda_i = f(h_1, c_{\alpha_1})$ or of the form $\lambda_i = g(h_1, h_2, \dots, h_n, c_{\alpha_1}, c_{\alpha_2}, \dots, c_{\alpha_n})$, do the different roots obtained by permutation of the $\alpha_1, \alpha_2, \dots, \alpha_n$ correspond to the order permutations of B in forming H ?

11-An application of matrix algebra to the study of human relations within-11 organizations - Katz

BIBLIOGRAPHY

Articles in Sociometry, 1938- , particularly the two immediately following.

Forsyth and Katz, "A matrix approach to the analysis of sociometric data: preliminary report," Sociometry, vol. 9 (1946).

Katz, "On the matrix analysis of sociometric data," Sociometry, vol. 10 (1947).

Festinger, "The analysis of sociograms using matrix algebra," Human Relations, vol. 2 (1949).

Luce and Perry, "A method of matrix analysis of group structure," Psychometrika, vol. 14 (1949).

Katz, "The distribution of the number of isolates in a group," Institute of Statistics Mimeo Series 36, (1950).