A MULTIDIMENSIONAL LOCAL LIMIT THEOREM FOR LATTICE DISTRIBUTIONS

by

Olaf Krafft

University of North Carolina
and
Technische Hochschule Karlsruhe

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Department of Statistics
University of North Carolina
Chapel Hill, N. C.

In this paper a local limit theorem for the extreme tail of a convolution of lattice distributed s-dimensional random vectors shall be proved. It is applied to multinomial distributions. The theorem is a generalization of a result obtained by Blackwell and Hodges [1] for the one-dimensional case and parallels a result obtained by Borovkov and Rogozin [2] for the case of absolutely continuously distributed s-dimensional random vectors.

Let $x' = (\xi_1 \xi_2, \dots, \xi_s)$ be a random vector in the s-dimensional Eucledian space R_s, a' be a fixed vector in R_s and H be a square matrix of order s with $h: = |\det H| > 0$.[1] A vector $k' = (\kappa_1, \kappa_2, \dots, \kappa_s) \in R_s$ is called an integer vector if all its components κ_j, $j = 1, 2, \dots, s$ are integers, positive or negative or zero. The set of all integer vectors k is denoted by K. A set

$$L_{a,H}: = \{ b \;:\; b = a + Hk,\ k \in K \}$$

is called a lattice with respect to a and H. L' is a sublattice of L if $b \in L'$ implies $b \in L$, and L'' is a proper sublattice of L if L'' is a sublattice of L and there is at least one $b \in L$ which is not in L''. A random vector x is said to be lattice-distributed (L-distributed) if there is a lattice L such that x is confined to L with probability one. In general this lattice L is not uniquely determined. But this can be rectified by the introduction of a concept analogous to that of the maximum span in the one-dimensional case, see Ranga Rao [6]:

For any L-distributed random vector x we say that its lattice L is maximal if

(i) $P(x \in L) = 1$,

(ii) there is no proper sublattice L'' of L such that $P(x \in L'') = 1$.

1) The notation "$\alpha := \beta$" is used when α is defined as equal to β.

It is shown by Heckendorff [3] that each non-degenerate lattice-distributed random vector has a maximal lattice.

In order to approximate the probabilities for the extreme tail of a convolution of lattice-distributed random vectors we will make the commonly used shift from the tails to the center. To this end let G denote the domain of convergence of the moment generating function M(z) of x,

$$M(z) := \sum_{\tilde{x}} \exp(z'\tilde{x})\, P(x = \tilde{x}).$$

By means of M(z) one can define a class of distributions P_z "conjugate" to the distribution P of x, see e.g. Khinchine [5],

$$P_z(x = \tilde{x}) := M^{-1}(z) \exp(z'\tilde{x})\, P(x = \tilde{x})$$

which are defined for all $z \in G$. Let F be an arbitrary non-void compact subset of the interior G^o of G in R_s and

$$\Phi := \{ b : b = \text{grad } \ln M(z),\ z \in F\}.$$

For $b \in \Phi$ we consider random vectors $y = y(b)$ which take values $\tilde{y} = \tilde{x} - b$ with probabilities

$$(1) \qquad P_{z(b)}(y = \tilde{y}) = M^{-1}(z(b)) \exp(z'(b)\tilde{x})\, P(x = \tilde{x})$$

where $z(b) \in F$ is that value of z for which $b = \text{grad}_{z(b)} \ln M(z)$. It may be noted that $z(b)$ is that value of z for which the function

$$m(z,b) := M(z) \exp(-z'b)$$

attains its minimum. As long as the distribution P is not concentrated on a hyperplane in R_s the vector $z(b)$ is uniquely determined by b and we can define

$$m(b) := M(z(b)) \exp(-z'(b)b),$$

see Hoeffding [4], Lemma 2.

In the following we will consider a sequence $x_1, x_2, \ldots$ of independent random vectors with the same distribution function. $\tilde{x}_{(n)}$ may be defined as $\tilde{x}_{(n)} := \sum_{j=1}^{n} \tilde{x}_j$ and only such values b will be of interest for which there is an $\tilde{x}_{(n)}$ such that $b_n := n^{-1} \tilde{x}_{(n)} = b$.

<u>Theorem:</u> Let $x_1, x_2, \ldots$ be a sequence of independent $L_{a,H}$-distributed random vectors with the same distribution function and let the following conditions be satisfied:

(A) The lattice $L_{a,H}$ is maximal,

(B) the determinant of the convariance matrix of x_1 is not equal to zero,

(C) $b_n \in \Phi$ for all n,

(D) the series $M(z + z(b_n)) := \sum_{\tilde{x}} \exp\{(z' + z'(b_n))\, \tilde{x}\} P(x_1 = \tilde{x})$ converges for all n and all z from a sphere with positive radius and 0 as origin.

Then the following asymptotic expansion is valid uniformly for all $b_n \in \Phi$:

$$(2) \quad P\left(\sum_{j=1}^{n} x_j = \tilde{x}_{(n)}\right) = \frac{m^n(b_n)\, h}{(2\pi n)^{s/2} \Delta^{1/2}(b_n)} \left\{ 1 + \sum_{j=1}^{\rho-1} c_j(b_n) n^{-j} + O(n^{-\rho}) \right\}.$$

Here $\Delta(bn)$ denotes the determinant of the covariance matrix of $y = y(b_n)$, the coefficients $c_j(b_n)$ are dependent only on the moments of y up to the order $(2j + 2)$ and ρ is any positive integer.

<u>Proof.</u> $P\left(\sum_{j=1}^{n} x_j = \tilde{x}_{(n)}\right) = \sum_{\tilde{x}_j : \Sigma \tilde{x}_j = \tilde{x}_{(n)}} \prod_{j=1}^{n} P(x_j = \tilde{x}_j) =$

$$= m^n(b_n) \sum_{\tilde{x}_j : \Sigma \tilde{x}_j = \tilde{x}_{(n)}} \exp\left\{-z'(b_n) \sum_{j=1}^{n} (\tilde{x}_j - b_n)\right\} \prod_{j=1}^{n} P(y_j = \tilde{x}_j - b_n) =$$

$$= m^n(b_n) \sum_{\tilde{x}_j : \Sigma(\tilde{x}_j - b_n) = 0} \prod_{j=1}^{n} P(y_j = \tilde{x}_j - b_n) = m^n(b_n)\, P\left(\sum_{j=1}^{n} y_j = 0\right).$$

For $P(\sum_{j=1}^{n} y_j = 0)$ we may apply the saddlepoint method used by Richer [8]. To that end we first note that if x_1 is $L_{a,H}$-distributed then $\tilde{x}_{(n)}$ will be of the form na + Hk. Hence $y(b_n)$ will be $L_{o,H}$-distributed. Condition (A) of theorem 2 in [8] can be replaced by our condition (A) as it is pointed out by Heckendorff [3]. For the expectation Ey we get

$$Ey = \sum_{\tilde{y}(b_n)} \tilde{y}(b_n)\, P(y = \tilde{y}(b_n))$$

$$= M^{-1}(z(b_n)) \{ \sum_{\tilde{\tilde{x}}_1} \tilde{x}_1 \exp(z'(b_n)\tilde{x}_1) P(x_1 = \tilde{x}_1) - b_n \sum_{\tilde{\tilde{x}}_1} \exp(z'(b_n)\tilde{x}_1) P(x_1 = \tilde{x}_1) \}$$

$$= M^{-1}(z(b_n)) \{ \text{grad}_{z(b_n)} M(z) - (\text{grad}_{z(b_n)} \ln M(z)) M(z(b_n)) \}$$

$$= M^{-1}(z(b_n)) \{ \text{grad}_{z(b_n)} M(z) - \text{grad}_{z(b_n)} M(z) \} = 0.$$

Since by (1) it is seen that $P_{z(b_n)}(y = \tilde{y}) = 0$ holds if and only if $P(x_1 = \tilde{x}_1) = 0$ condition (B) implies that $\Delta(b_n) \neq 0$ for all b_n.

The proof for that

$$P(\sum_{j=1}^{n} y_j = 0) = \frac{h}{(2\pi n)^{s/2} \Delta^{1/2}(b_n)} \{ 1 + \sum_{j=1}^{\rho-1} c_j(b_n)\, n^{-j} + O(n^{-\rho}) \}$$

holds is then with slight modifications the same as it is given by Richter [8]. The saddlepoint will be at $z_o = 0$, the expansions of the logarithm of the moment generating function of $y(b_n)$ in Taylor series will be uniform for $b_n \in \Phi$. It should be mentioned here that the coefficients of the expansions in Richter's papers [7] and [8] are partially incorrect. So in formula (11) in [7] the factor $\rho_3^{\,2}(3!)^{-2}$ should be multiplied by 1/2 - the same correction should be made in [8], p. 102, line 8 f.b. - and the formula for I_1 in the second line from the top, p. 214 in [7], becomes

$$I_1 = \frac{i\sqrt{2\pi}\, e^{n(K(z_o) - z_o'\tau\sigma)}}{\sqrt{n\, K''\,(z_o)}} \left[1 + \{\frac{\rho_4(z_o)}{8} - \frac{5}{24}\; \rho_3^2(z_o)\}\frac{1}{n} + o(\frac{1}{n}) \right].$$

Whereas in applications of this theorem to special cases the evaluation of $z(b_n)$ and of $\Delta(b_n)$ might be rather cumbersome it can easily be done for multinomial distributions:

Let x be an s-dimensional random vector and e_j, $j = 1, 2, \ldots, s$, be the unit vectors in R_s. The distribution of x may be given by

$$P\,(x = e_j) = p_j,\ j = 1, 2, \ldots, s,\quad P\,(x = 0) = p_{s+1},\ \sum_{j=1}^{s+1} p_j = 1,\ p_j > 0,\ j=1,2,\ldots,s+1.$$

x is then lattice-distributed with maximal lattice $L_{o,I}$ where I is the s-dimensional unit matrix, h = 1. The sum of n such vectors x defines an (s + 1)-dimensional multinomial distribution, i. e., if there are (s + 1) pairwise disjoint events B_j as results of an experiment with $P\,(B_j) = p_j$, $j = 1, 2, \ldots, s + 1$, and $p\,(n_1, n_2, \ldots, n_{s+1})$ is the probability that in n independent repetitions of the experiment the event B_1 occurs n_1 times, B_2 occurs n_2 times,..., B_{s+1} occurs n_{s+1} times, $\sum_{j=1}^{s+1} n_j = n$, $n_j \geq 0$, $j = 1, a, \ldots, s + 1$, then

$$p(n_1, n_2, \ldots, n_{s+1}) = P\,(\sum_{j=1}^{n} x_j = \sum_{j=1}^{s} n_j\, e_j) = \frac{n\,!}{n_1!\; n_2!\; \ldots n_{s+1}!}\; p_1^{n_1} p_2^{n_2} \ldots p_{s+1}^{n_{s+1}}.$$

For the moment generating function M(z) of x we get with $z' = (\zeta_1, \zeta_2, \ldots \zeta_s)$

$$M(z) = p_{s+1} + \sum_{j=1}^{s} e^{\zeta_j}\, p_j\ .$$

Let $n^{-1} \sum_{j=1}^{n} \tilde{x}_j' = b_n' = (\beta_1, \beta_2, \ldots \beta_s)$ and consider for fixed $\epsilon > 0$ values b_n' with $\beta_j > \epsilon$, $j = 1, 2, \ldots, s$, $\beta_{s+1} := 1 - \sum_{j=1}^{s} \beta_j < 1 - \epsilon$. The conditions of the theorem are then satisfied. The vector $z(b_n)$ can be calculated differentiating with respect to z the function

$$M(z)\ \exp\,(z'b_n) = p_{s+1}\exp\,(\,-\sum_{j=1}^{s} \zeta_j\, \beta_j) + \sum_{j=1}^{s} p_j \exp(\zeta_j(1-\beta_j) - \sum_{k\neq j} \zeta_k\beta_k).$$

The partial derivatives will be zero for those values of z for which

$$\beta_j \, (p_{s+1} + \sum_{k=1}^{s} p_k e^{\zeta_k}) = p_j e^{\zeta_j}, \; j = 1, 2, \ldots, s, \text{ i. e., if}$$

$$\zeta_j = \ln \, (\beta_j \, p_{s+1} \, / \, p_j \quad \beta_{s+1}) \; .$$

This result makes it possible to calculate the functions $M\,(z + z(b_n))$ and $m\,(b_n)$ to

$$M\,(z + z(b_n)) = \frac{p_{s+1}}{\beta_{s+1}} \, \{\beta_{s+1} + \sum_{j=1}^{s} \beta_j e^{\zeta_j}\} \quad \text{and}$$

$$m\,(b_n) = \prod_{j=1}^{s+1} \, (p_j/\beta_j)^{\beta_j} \, .$$

$y\,(b_n)$ takes values $e_j - b_n$, $j = 1, 2, \ldots, s$, and $(-b_n)$ with probabilities $P(y = = e_j - b_n) = \beta_j$, $j = 1, 2, \ldots, s$, $P(y = -b) = \beta_{s+1}$, from which the covariance matrix $C(b_n)$ is calculated as

$$C(b_n) = (c_{ij}), c_{ij} = \begin{cases} -\beta_i \beta_j, & i \neq j \\ (1-\beta_i)\,\beta_i, & i = j. \end{cases}$$

Adding all remaining rows of $C(b_n)$ to the first one we get for $\Delta(b_n)$

$$\Delta(b_n) = \beta_{s+1} \begin{vmatrix} \beta_1, & \beta_2, & \ldots, & \beta_s \\ -\beta_1\beta_2, & (1-\beta_2)\beta_2, & \ldots, & -\beta_1\beta_s \\ \vdots & \vdots & & \vdots \\ -\beta_1\,\beta_s, & -\beta_2\,\beta_s, & \ldots, & (1-\beta_s)\beta_s \end{vmatrix} .$$

If we now add the first row, multiplied with β_j, to the j-th row, $j = 2, 3, \ldots, s$, we get

$$\Delta(b_n) = \prod_{j=1}^{s+1} \beta_j .$$

With the help of (2) we can then estimate the probabilities (3) to

(4) $$p(n_1, n_2, \ldots, n_{s+1}) = \{(2\pi n)^s \prod_{j=1}^{s+1} (n_j/n)\}^{-1/2} \prod_{j=1}^{s+1} (np_j/n_j)^{n_j} \{1+O(n^{-1})\},$$

an expression which is obtained also by Hoeffding [4], applying Sterling's formula to (3). The function m(b) is closely related to the function I (A(s), F) used by Hoeffding to derive more general results on probabilities of large deviations.

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