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ON A CLASS OF ALIGNED RANK ORDER TESTS FOR MULTIRESPONSE EXPERIMENTS IN SOME INCOMPLETE BLOCK DESIGNS

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1. Summary and introduction. Consider n replications of an incomplete block design D consisting of b blocks of constant size $k(\geq 2)$ to which $v(\geq k)$ treatments are applied in such a way that (i) no treatment occurs more than once in any block, (ii) the jth treatment occurs in $r_j(\leq b)$ blocks (j=1,...,v), and (iii) the (j, j')th treatments occur together in $r_{jj'}$ (>0) blocks ($j \neq j'=1,\ldots,v$). [Conventionally, we let $r_{jj} = r_j$, j=1,...,v.] Then, by definition

(1.1) $bk = \sum_{j=1}^{v} r_j$ and $bk(k-1) = \sum_{j\neq j'=1}^{v} r_{jj'}$, $(\Rightarrow bk^2 = \sum_{j,j'=1}^{v} r_{jj'})$.

Let then S_i stand for the set of treatments occuring in the ith block, i=1,...,b. [Thus, $S_1,\ldots,S_b$ are all subsets of (1,...,v).] For the αth replicate, the response of the plot in the ith block and receiving the jth treatment is a stochastic p-vector $\underset{\sim}{X}_{\alpha ij}$ and is expressed as

(1.2) $\underset{\sim}{X}_{\alpha ij} = \underset{\sim}{\mu}_\alpha + \underset{\sim}{\beta}_{\alpha i} + \underset{\sim}{\tau}_j + \underset{\sim}{\varepsilon}_{\alpha ij}$, $j \in S_i$, i=1,...,b, α=1,...,n,

where the $\underset{\sim}{\mu}_\alpha$ and the $\underset{\sim}{\beta}_{\alpha i}$ are respectively the replicate and block effects (nuisance parameters in 'fixed effect' model or spurious

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random variables in 'mixed effect' model), $\tau_1,\ldots,\tau_v$ are the treatment effects (parameters of interest) and the $\varepsilon_{\alpha ij}$ are the error vectors. It is assumed that $[\varepsilon_{\alpha ij},\ j \in S_i]$ have jointly a continuous cumulative distribution function (cdf) $G(x_1,\ldots,x_k)$ which is symmetric in its k vectors. This includes the conventional assumption of independence and identity of the distributions of $\varepsilon_{\alpha ij},\ j \in S_i$, $i=1,\ldots,b$ as a special case. We may, without any loss of generality, set $\sum_{j=1}^{v} \tau_j = 0$, and state the null hypothesis of no treatment effects as

(1.3) $H_o:\ \tau_1 = \cdots = \tau_v = 0$ vs $H_1:\ \tau_j \neq 0$ for at least one $j(=1,\ldots,v)$.

In the univariate case (i.e., p=1), intra-block rank tests for this problem are due to Durbin (1951), Benard and Elteren (1953), and Bhapkar (1961), among others. For some special balanced designs, the studies made by Elteren and Noether (1959) and Bhapkar (1963) reveal the low efficiency of these tests, particularly when k is small. In complete block designs, it is shown [cf. Sen (1968)] that the use of ranking after alignment increases the efficiency of the rank tests. In the present paper, the theory developed in Sen (1968, 1969) is extended to the case of incomplete block designs (as has been sketched above). Since the aligned rank tests are not genuinely distribution-free, even when p=1, [cf. Sen (1968)], our first task is to show that these are conditionally distribution-free. This has been accomplished here by extending the permutation invariance structure, developed in Sen (1968), to a wider group. Section 2 of this paper is devoted to the preliminary notions and definitions. In section 3, the permutational invariance structure of the sample observations is studied. The next two sections are devoted to the study of the large sample (permutational as well as unconditional) distribution theory of the porposed statistics. The last section is devoted to the asymptotic power and efficiency of the proposed tests.

2. Preliminary notions. We define the aligned observations by

(2.1) $\underset{\sim}{Y}_{\alpha ij} = \underset{\sim}{X}_{\alpha ij} - k^{-1} \Sigma_{\ell \varepsilon S_i} \underset{\sim}{X}_{\alpha i\ell}$, $j \varepsilon S_i$, $i=1,\ldots,b$, $\alpha=1,\ldots,n$.

Also, let

(2.2) $\underset{\sim}{\tau}_{j,i} = \underset{\sim}{\tau}_j - k^{-1} \Sigma_{\ell \varepsilon S_i} \underset{\sim}{\tau}_\ell$ and $\underset{\sim}{e}_{\alpha ij} = \underset{\sim}{\varepsilon}_{\alpha ij} - k^{-1} \Sigma_{\ell \varepsilon S_i} \underset{\sim}{\varepsilon}_{\alpha i\ell}$, $j \varepsilon S_i$,

for $i=1,\ldots,b$, $\alpha=1,\ldots,n$. Then, from (1.2), (2.1) and (2.2), we have

(2.3) $\underset{\sim}{Y}_{\alpha ij} = \underset{\sim}{\tau}_{j,i} + \underset{\sim}{e}_{\alpha ij}$, $j \varepsilon S_i$, $i=1,\ldots,b$, $\alpha=1,\ldots,n$.

Since $G(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_k)$ is assumed to be symmetric in $\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_k$, it follows readily that $F(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_k)$, the joint cdf of $\underset{\sim}{e}_{\alpha ij}$, $j \varepsilon S_i$, is also symmetric in its k argument vectors. Thus, if $\mathcal{F}_{kp}$ stands for the class of all $G(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_k)$ which are symmetric in their k (p-)vectors, then

(2.4) $F(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_k) \varepsilon \mathcal{F}_{kp}$ whenever $G(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_k) \varepsilon \tilde{\mathcal{F}}_{kp}$.

Let now N = nbk, and consider p sequences of general rank scores

(2.5) $E^{(t)}_{N,s} = J_{N,t}(s/(N+1)),\ 1\le s\le N,$ for $t=1,\ldots,p$,

where for each t, $J_{N,t}$ satisfies the regularity conditions (c·1)-(c·5) of Sen (1967), which are adapted from Chernoff and Savage (1958). [Thus, $\lim_{N\to\infty} J_{N,t}(u) = J_t(u)$ exists for all $0<u<1$ and is not a constant.] For the t-th variate, we let $R^{(t)}_{\alpha ij}$ be the aligned rank of $Y^{(t)}_{\alpha ij}$ among the N observations $Y^{(t)}_{\alpha ij}$, $j \varepsilon S_i$, $i=1,\ldots,b$, $\alpha=1,\ldots,n$. For notational simplicity, we let

(2.6) $\eta^{(t)}_{\alpha ij} = E^{(t)}_{N,R^{(t)}_{\alpha ij}}$, $j \varepsilon S_i$, $i=1,\ldots,b$, $\alpha=1,\ldots,n$;

(2.7) $\eta^{(t)}_{\alpha i\cdot} = k^{-1} \Sigma_{j\varepsilon S_i} \eta^{(t)}_{\alpha ij}$, $\eta^{(t)}_{\alpha\cdot\cdot} = b^{-1} \sum_{i=1}^{b} \eta^{(t)}_{\alpha i\cdot}$ and $\eta^{(t)}_{\cdot\cdot\cdot} = n^{-1} \sum_{\alpha=1}^{n} \eta^{(t)}_{\alpha\cdot\cdot}$,

for $t=1,\ldots,p$. Our proposed tests are based on the statistics.

(2.8) $T^{(t)}_{N,j} = \frac{1}{n} \sum_{\alpha=1}^{n} \Sigma_{i\varepsilon P_j} \eta^{(t)}_{\alpha ij}$, $j=1,\ldots,v$, $t=1,\ldots,p$,

where

(2.9) $P_j = \{i : j \varepsilon S_i\}$, $j=1,\ldots,v$.

For later use, we define

(2.10) $v^{(1)}_{N,tt'} = \frac{1}{nbk} \sum_{\alpha=1}^{n} \sum_{i=1}^{b} \sum_{j\varepsilon S_i} [\eta^{(t)}_{\alpha ij} - \eta^{(t)}_{\alpha i\cdot}]\, [\eta^{(t')}_{\alpha ij} - \eta^{(t')}_{\alpha i\cdot}]$,

for $t, t'=1,\ldots,p$, and

(2.11) $v^{(2)}_{N,tt'} = \frac{1}{nb} \sum_{\alpha=1}^{n} \sum_{i=1}^{b} [\eta^{(t)}_{\alpha i\cdot} - \eta^{(t)}_{\alpha\cdot\cdot}]\, [\eta^{(t')}_{\alpha i\cdot} - \eta^{(t')}_{\alpha\cdot\cdot}]$, for $t,t'=1,\ldots,p$;

(2.12) $\underset{\sim}{V}_N^{(t)} = ((v_{N,tt'}^{(1)}))$ and $\underset{\sim}{V}_N^{(2)} = ((v_{N,tt'}^{(2)}))$.

Further, let

(2.13) $\underset{\sim}{A}^{(1)} = ((a_{jj'}^{(1)}))$ and $\underset{\sim}{A}^{(2)} = ((a_{jj'}^{(2)}))$

where

(2.14) $a_{jj'}^{(1)} = [kr_j\delta_{jj'} - r_{jj'}]/(k-1)$, $j,j'=1,\ldots,v$;

(2.15) $a_{jj'}^{(2)} = [br_{jj'} - r_j r_{j'}]/(b-1)$, $j,j'=1,\ldots,v$,

and $\delta_{jj'}$ is the usual Kronecker delta. Finally, let

(2.16) $\underset{\sim}{W}_N = \underset{\sim}{A}^{(1)} \otimes \underset{\sim}{V}_N^{(1)} + \underset{\sim}{A}^{(2)} \otimes \underset{\sim}{V}_N^{(2)}$,

where $\otimes$ stands for the Kronecker product of two matrices. Our basic assumption on the design matrices $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ is the following:

(2.17) Rank of $\underset{\sim}{A}^{(1)} = v-1$; $\underset{\sim}{A}^{(2)}$ and $b\underset{\sim}{A}^{(1)}-(b-1)\underset{\sim}{A}^{(2)}$ are non-negative definite

[Without going into the details and referring to Kempthorne (1952, Chapters 26 and 27), we may verify that (2.17) holds for the entire class of balanced and partially balanced incomplete block designs. This also holds for the class of group divisible designs.] Throughout the paper, it will be assumed that $k > 2$; for $k = 2$, the problem reduces to that of the multivariate paired comparisons [considered by Sen and David (1968) and Puri and Shane (1969), among others], and hence, will not be considered here. Finally, we assume that both $\underset{\sim}{V}_N^{(1)}$ and $\underset{\sim}{V}_N^{(2)}$ are positive definite, in probability; the justification for this assumption will follow from results to be considered later on in section 4.

3. Permutational invariance and the test statistic. We denote by $\underset{\sim}{Y}_{\alpha i} = (\underset{\sim}{Y}_{\alpha ij},\ j \in S_i)$, $\underset{\sim}{Y}_{\alpha} = (\underset{\sim}{Y}_{\alpha 1},\ldots, \underset{\sim}{Y}_{\alpha b})$, $\alpha=1,\ldots,n$, and $\underset{\sim}{Y}_N^o = (\underset{\sim}{Y}_1,\ldots, \underset{\sim}{Y}_n)$. Under H_o in (1.3), $\underset{\sim}{Y}_{\alpha ij} = \underset{\sim}{e}_{\alpha ij}$ for all α, i and j. Hence, by (2.4), (i) the joint distribution of $\underset{\sim}{Y}_{\alpha i}$ remains invariant under any permutation of the k vectors among themselves (there being k! such permutations), $i=1,\ldots,b$, and (ii) as $\underset{\sim}{Y}_{\alpha i}$, $i=1,\ldots,b$ are independent and identically distributed, the joint distribution of $\underset{\sim}{Y}_{\alpha}$ remains invariant under any permutation of the b sets $\underset{\sim}{Y}_{\alpha i}$, $i=1,\ldots,b$, among themselves (there being b! such permutations). Consider now the group $\mathcal{G}$ of $b!(k!)^b$ transformations $\{g\}$ where

$$(3.1)\quad g\underset{\sim}{Y}_{\alpha} = \underset{\sim}{Y}_{\alpha}^{*} \quad (g \in \mathcal{G})$$

is such that (i) $\underset{\sim}{Y}_{\alpha ij}^{*}$, $j \in S_i$ is any permutation of $\underset{\sim}{Y}_{\alpha ij}$, $j \in S_i$, and (ii) $(\underset{\sim}{Y}_{\alpha 1}^{*},\ldots, \underset{\sim}{Y}_{\alpha b}^{*})$ is any permutation of $(\underset{\sim}{Y}_{\alpha 1},\ldots, \underset{\sim}{Y}_{\alpha b})$. Clearly $\mathcal{G}$ maps the sample space of $\underset{\sim}{Y}_{\alpha}$ onto itself and leaves its distribution invariant when (1.3) holds. Define now a compound group $\mathcal{G}_n$ of transformations $\{g_n\}$ by

$$(3.2)\quad g_n\underset{\sim}{Y}_N^o = \underset{\sim}{Y}_N^{o*} = [g_n^{(1)}\underset{\sim}{Y}_1,\ldots, g_n^{(n)}\underset{\sim}{Y}_n],\ g_n^{(\alpha)} \in \mathcal{G},\ \alpha=1,\ldots,n.$$

Then $\mathcal{G}_n$ contains $[b!(k!)^b]^n$ transformations, and under (1.3), it leaves the joint distribution of $\underset{\sim}{Y}_N^o$ invariant. Let

(3.3) $S(\underset{\sim}{Y}_N^o) = \{\underset{\sim}{Y}_N^{o*} = g_n \underset{\sim}{Y}_N^o : g_n \in \mathcal{G}_n\}$.

Then, it follows from the above discussion that

(3.4) $P\{\underset{\sim}{Y}_N^o = \underset{\sim}{Y}_N^{o*} \mid S(\underset{\sim}{Y}_N^o),\ H_o \text{ in } (1.3)\} = 1/N^*;\ N^* = (b!(k!)^b)^n,$

for all $\underset{\sim}{Y}_N^{o*} \in S(\underset{\sim}{Y}_N^o)$, whenever $G \in \mathcal{F}_{kp}$. Let us denote this conditional (permutational) probability measure by $\mathcal{P}_n$. As $\mathcal{P}_n$ is completely specified by (3.4), the existence of similar (conditional) tests follow on the same line as in Hoeffding (1952, pp. 169-171). We shall construct such tests based on the statistics in (2.8). It may be noted that we could have worked with the conditional probability measure over the $(k!)^{nb}$ intra-block permutations, as has been considered for the complete block designs in Sen (1968). However, for incomplete block designs, this would result in a permutation covariance matrix of the statistics which would be more complicated and operationally more difficult to manipulate.

Now, it follows from (3.4) and some standard computations that

(3.5) $E_{\mathcal{P}_n}\{\eta_{\alpha ij}^{(t)}\} = \eta_{\alpha\cdot\cdot}^{(t)}$, $j \in S_i$, $i=1,\ldots,b$; $\alpha=1,\ldots,n$;

(3.6) $E_{\mathcal{P}_n}\{\eta_{\alpha ij}^{(t)} \cdot \eta_{\alpha ij'}^{(t')}\} = \frac{1}{bk(k-1)} \sum_{i=1}^{b} \sum_{j \neq j' \in S_i} \eta_{\alpha ij}^{(t)} \eta_{\alpha ij'}^{(t')}$, $j \neq j' \in S_i$,

(3.7) $E_{\mathcal{P}_n}\{\eta_{\alpha ij}^{(t)} \cdot \eta_{\alpha ij}^{(t')}\} = \frac{1}{bk} \sum_{i=1}^{b} \sum_{j \in S_i} \eta_{\alpha ij}^{(t)} \eta_{\alpha ij}^{(t')}$, $j \in S_i$, $i=1,\ldots,b$;

(3.8) $E_{\mathcal{P}_n}\{\eta^{(t)}_{\alpha ij}\cdot\eta^{(t')}_{\alpha i'j'}\} = \frac{1}{b(b-1)} \sum_{i\neq i'=1}^{b} \eta^{(t)}_{\alpha i\cdot}\eta^{(t')}_{\alpha i'\cdot}$ for $j \varepsilon S_i$, $j' \varepsilon S_{i'}$, $i\neq i'=1,\ldots,b$, $\alpha = 1,\ldots,n$. This leads (after some simplifications) to

(3.9) $E_{\mathcal{P}_n}\{T^{(t)}_{N,j}\} = r_j\eta^{(t)}_{\cdots}$ for $j=1,\ldots,v$, $t=1,\ldots,p$.

(3.10) $\mathrm{COV}_{\mathcal{P}_n}\{T^{(t)}_{N,j}, T^{(t')}_{N,j'}\} = a^{(1)}_{jj'}v^{(1)}_{n,tt'} + a^{(2)}_{jj'}v^{(2)}_{n,tt'}$,

$j,j'=1,\ldots,v$; $t,t'=1,\ldots,p$, where the $a^{(s)}_{jj'}$ and $v^{(s)}_{n,tt'}$ are defined by (2.10) through (2.15). Moreover, by definition in (2.8),

(3.11) $\sum_{j=1}^{v} T^{(t)}_{N,j} = (bk)\eta^{(t)}_{\cdots}$ for all $t=1,\ldots,p$.

Thus, at most $(v-1)p$ of the vp statistics in (2.8) are linearly independent. Let now $\underset{\sim}{T}_N = (T^{(1)}_{N,1},\ldots,T^{(1)}_{N,v},\ldots,T^{(p)}_{N,1},\ldots,T^{(p)}_{N,v})$. Then from (3.9) and (3.10) we obtain that

(3.12) $E_{\mathcal{P}_n}\{\underset{\sim}{T}_N\} = \underset{\sim}{\tilde{\eta}} = (r_1\eta^{(1)}_{\cdots},\ldots,r_v\eta^{(1)}_{\cdots},\ldots,r_1\eta^{(p)}_{\cdots},\ldots,r_v\eta^{(v)}_{\cdots})$;

(3.13) $nV_{\mathcal{P}_n}(\underset{\sim}{T}_N) = \underset{\sim}{W}_N$,

where $\underset{\sim}{W}_N$ is defined by (2.16). Now, by definition, $\underset{\sim}{A}^{(1)}$ and $\underset{\sim}{A}^{(2)}$ are both positive semi-definite, and by (2.17) $A^{(1)}$ is of rank $v-1$. Thus, if $\underset{\sim}{V}^{(1)}_N$ is of full-rank, the rank of $\underset{\sim}{W}_N$ is $p(v-1)$. In fact, if Rank $\underset{\sim}{V}^{(1)}_N =$ $p' =$ Rank $\underset{\sim}{V}^{(2)}_N$, the rank of $\underset{\sim}{W}_N$ is $p'(v-1)$. Now, either we can work with a subvector of $\underset{\sim}{T}_N$ with $p'(v-1)$ elements for which the corresponding co-factor of $\underset{\sim}{W}_N$ is positive definite, construct a quadratic form in these variables with the inverse of their permutation covariance matrix as the discriminant and finally symmetrize it on using (3.11), or we may work with the generalized inverse of $\underset{\sim}{W}_N$ and construct a quadratic form in $\underset{\sim}{T}_N - \tilde{\underset{\sim}{\eta}}$. For notational simplicity, we consider the second form and write the test statistic as

(3.14) $\mathcal{L}_N = n[\underset{\sim}{T}_N - \tilde{\underset{\sim}{\eta}}]\ \underset{\sim}{W}_N^*\ [\underset{\sim}{T}_N - \tilde{\underset{\sim}{\eta}}]'$,

where $\underset{\sim}{W}_N^*$ is the generalized inverse of $\underset{\sim}{W}_N$.

For small values of n, b and k, the exact permutation distribution of $\mathcal{L}_N$ can be obtained by considering the $N^*[=(b!(k!)^b)^n]$ conditionally equally likely permutations obtained by applying the group of transformations $\mathcal{G}_n$ on $\underset{\sim}{Y}_N^o$. Thus, a conditionally distribution-free test for H_o in (1.3) can be constructed [whatever be $G \in \mathcal{F}_{kp}$]. This procedure becomes prohibitively laborious for large sample sizes. For this reason, we consider first the large sample permutation distribution of $\mathcal{L}_N$.

4. Asymptotic permutation distribution of $\mathcal{L}_N$. Corresponding to the pk-variate cdf $F(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_k)$, we define (1) the marginal cdf of the t-th variate jth treatment observation by $F_{[t]}(x)$ [by (2.4), it does

not depend on j], (ii) the bivariate cdf of the (t,t')th variates, jth treatment observations by $F^{(1)}_{[t,t']}(x,y)$ ($t \neq t'$) and (iii) the bivariate cdf of t-th variate jth treatment and t'-th variate j'-th treatment observations by $F^{(2)}_{[t,t']}(x,y)$ for $j \neq j'=1,\ldots,v$, $t,t'=1,\ldots,p$. Let then

$$(4.1)\quad \nu^{(i)}_{tt'} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_t[F_{[t]}(x)]\, J_{t'}[F_{[t']}(y)]\, dF^{(i)}_{[t,t']}(x,y) - \mu_t\mu_{t'}\ ,$$

$i=1,2$, for $t,t'=1,\ldots,p$, where $\mu_t = \int_0^1 J_t(u)du$, $t=1,\ldots,p$, and $F^{(1)}_{[t,t]}(x,y)$ reduces to the univariate cdf $F_{[t]}(x)$, as $x=y$ a e..

Further let

$$(4.2)\quad \nu^{(i)} = ((\nu^{(i)}_{tt'}))_{t,t'=1,\ldots,p}\ ,\ i=1,2;$$

$$\underset{\sim}{A} = [(k-1)/k]\ \underset{\sim}{A}^{(1)} + [(b-1)/bk]\ \underset{\sim}{A}^{(2)},$$

$$(4.3)\quad \underset{\sim}{B} = [(k-1)/k]\underset{\sim}{A}^{(1)} - [(b-1)(k-1)/bk]\underset{\sim}{A}^{(2)}\ ;$$

$$(4.4)\quad \underset{\sim}{\Omega} = \underset{\sim}{A} \otimes \underset{\sim}{\nu}^{(1)} - \underset{\sim}{B} \otimes \underset{\sim}{\nu}^{(2)}\ .$$

<u>Theorem 4.1</u>. <u>Under</u> H_o <u>in</u> (1.3), $\underset{\sim}{W}_N$, <u>defined by</u> (2.16), <u>converges in probability</u> (<u>as</u> $n \to \infty$) <u>to</u> $\underset{\sim}{\Omega}$, <u>defined by</u> (4.4).

<u>OUTLINE OF THE PROOF</u>. Proceeding precisely on the same line as in lemma 4.2 of Sen (1967), following some standard analysis and omitting the details, we obtain that under H_o in (1.3)

(4.5) $\underset{\sim}{V}_N^{(1)} \overset{p}{\to} [(k-1)/k] [\underset{\sim}{\nu}^{(1)} - \underset{\sim}{\nu}^{(2)}]$,

(4.6) $\underset{\sim}{V}_N^{(2)} \overset{p}{\to} [(b-1)/bk] [\underset{\sim}{\nu}^{(1)} + (k-1)\underset{\sim}{\nu}^{(2)}]$.

The rest of the proof follows from (2.16), (4.3), (4.4), (4.5) and (4.6).

For the complete block design (in the multiresponse case), conditions under which $\underset{\sim}{\nu}^{(1)} - \underset{\sim}{\nu}^{(2)}$ is positive definite are studied in theorem 4.1 of Sen (1969). The same conditions apply for the incomplete block designs under consideration. Hence, $\underset{\sim}{V}_N^{(1)}$ is positive definite, in probability. Also, by a direct multivariate generalization of lemma 4.4 of Sen (1968) it follows that $\underset{\sim}{\nu}^{(1)} + (k-1)\underset{\sim}{\nu}^{(2)}$ is positive semi-definite, and hence, $\underset{\sim}{V}_N^{(2)}$ is positive semi-definite, in probability. Thus, from (2.17), (4.3), (4.4), (4.5) and (4.6), it follows that under the conditions of theorem 4.1 of Sen (1969), $\underset{\sim}{\Omega}$ is of rank p(v-1), and hence, $\underset{\sim}{W}_N$ is of rank p(v-1), in probability (as $n \to \infty$).

Theorem 4.2. Under the conditions (c.1)-(c.4) of Sen (1967) and of theorem 4.1 of Sen (1969), the conditional (permutational) distribution of $\mathcal{L}_N$ (under $\mathcal{P}_n$) converges, in probability, to a chi-square distribution with p(v-1) degrees of freedom (d.f.).

OUTLINE OF THE PROOF. By virtue of (3.12), (3.13) and (3.14), the theorem would follow if we can show that the permutation distribution of $n^{1/2}(\underset{\sim}{T}_N - \underset{\sim}{\tilde{\eta}})$ converges, in probability, to a multinormal distribution of rank p(v-1). For this, let $\underset{\sim}{\delta} = (\delta_1, \ldots, \delta_v)$ $(\neq \underset{\sim}{0})$ be any real vector $(\perp \underset{\sim}{\ell}_v)$. Then, we are to show that under $\mathcal{P}_n$, the distribution of

$$(4.7)\quad \underset{\sim}{Z}_n = [n^{1/2} \sum_{j=1}^{v} \delta_j (T_{N,j}^{(t)} - r_j \eta_{\ldots}^{(t)}),\ t=1,\ldots,p]$$

converges (in probability,) to a p-variate normal distribution. Now,

$$(4.8)\quad \underset{\sim}{Z}_n = n^{-1/2} \sum_{\alpha=1}^{n} \underset{\sim}{Z}_{n\alpha};\quad \underset{\sim}{Z}_{n\alpha} = (Z_{n\alpha}^{(1)}, \ldots, Z_{n\alpha}^{(p)}),\ \alpha=1,\ldots,n,$$

where

$$(4.9)\quad Z_{n\alpha}^{(t)} = [\sum_{j=1}^{v} \delta_j \Sigma_{i\varepsilon P_j} (\eta_{\alpha ij}^{(t)} - \eta_{\alpha..}^{(t)})]/bk,\ \alpha=1,\ldots,n,\ t=1,\ldots,p.$$

Since, under $\mathcal{P}_n$, $\underset{\sim}{Z}_{n1}, \ldots, \underset{\sim}{Z}_{nn}$ are mutually independent, the proof of the asymptotic normality of $\underset{\sim}{Z}_n$ follows by using the central limit theorem and proceeding as in the proof of theorem 3.2 of Sen(1968).

It follows from theorem 4.2 that for large n, the critical values of $\mathcal{L}_N$ can be approximated by those of the chi-square distribution with p(v-1) d.f.

5. <u>Asymptotic non-null distribution of</u> $\mathcal{L}_N$. The asymptotic distribution of $\mathcal{L}_N$ depends on the asymptotic joint distribution of the $T_{N,j}^{(t)}$.

An easy way to derive the asymptotic distribution of the $T_{N,j}^{(k)}$ is to consider initially the set of bkp random variables

$$(5.1)\quad U_{n,ij}^{(t)} = n^{-1} \sum_{\alpha=1}^{n} \eta_{\alpha ij}^{(t)} \text{ for } j \varepsilon S_i,\ i=1,\ldots,b \text{ and } t=1,\ldots,p,$$

to derive their joint distribution, and finally using the fact that $T_{N,j}^{(t)} = \Sigma_{i\varepsilon P_j} U_{n,ij}^{(t)}$, $j=1,\ldots,v$, $t=1,\ldots,p$, to deduce the joint distribution of the later variables. The asymptotic normality of the joint distributions of the variables in (5.1) follows precisely on the same line as in theorem 5.1 of Sen (1967), with further generalizations in theorem 5.1 of Sen (1969). We denote by $F_{ij[t]}(x)$ the marginal cdf of $Y_{\alpha ij}^{(t)}$, defined by (2.1), and let

$$(5.2)\quad H_{[t]}(x) = \frac{1}{bk}\sum_{i=1}^{b} \Sigma_{j\varepsilon S_i} F_{ij[t]}(x),\ t=1,\ldots,p.$$

Further, let $F_{ij,i'j'[t,t']}(x,y)$ be the joint cdf of $(Y_{\alpha ij}^{(t)}, Y_{\alpha i'j'}^{(t)})$, for $j \varepsilon S_i$, $j' \varepsilon S_{i'}$, $i,i'=1,\ldots,b$; $t,t'=1,\ldots,p$. By definition for $i\neq i'$, $F_{ij,i'j'[t,t']}(x,y) = F_{ij[t]}(x)F_{i'j'[t']}(y)$. Let then

$$(5.3)\quad \mu_{n,ij}^{(t)} = \int_{-\infty}^{\infty} J_t[H_{[t]}(x)]dF_{ij[t]}(x),\ j \varepsilon S_i,\ i=1,\ldots,b;\ t=1,\ldots,p.$$

Also, let

$$(5.4)\quad \beta_{ij,ij;rs,r's'}^{(t,t)} = \iint_{-\infty<x<y<\infty} F_{ij[t]}(x)[1-F_{ij[t]}(y)]J_t'[H_{[t]}(x)]J_t'[H_{[t]}(y)]\cdot \{dF_{rs[t]}(x)dF_{r's'[t]}(y) + dF_{rs[t]}(y)dF_{r's'[t]}(x)\}$$

for $s(s') \varepsilon S_r(S_{r'})$; r, $r'=1,\ldots,b$ and $j \varepsilon S_i$, $i=1,\ldots,b$, $t=1,\ldots,p$, and for $(i,j,t) \neq (i',j',t')$, we let

$$(5.5)\quad \beta_{ij,i'j';rs,r's'}^{(t,t')} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{F_{ij,i'j'[t,t']}(x,y)-F_{ij[t]}(x)F_{i'j'[t']}(y)\}\ .$$

$$J_t'[H_{[t]}(x)]J_{t'}'[H_{[t']}(y)]dF_{rs[t]}(x)dF_{r's'[t']}(y)$$

Let then

(5.6) $$\beta^{*(t,t')}_{ij,i'j'} = \{\sum_{r=1}^{b}\sum_{r'=1}^{b}\sum_{s\varepsilon S_r}\sum_{s'\varepsilon S_{r'}}[\beta^{(t,t')}_{ij,i'j';rs,r's'} + \beta^{(t,t')}_{rs,r's';ij,i'j'} - \beta^{(t,t')}_{ij,r's';rs,i'j'} - \beta^{(t,t')}_{rs,i'j';ij,r's'}]\} \;/\; (bk)^2 \;,$$

for $t,t'=1,\ldots,p$, $j \varepsilon S_i$, $j' \varepsilon S_{i'}$; $i,i'=1,\ldots,b$. Then from theorem 5.1 of Sen (1967) and some simple (but somewhat lengthy) computations, we arrive at the following.

Theorem 5.1. Under the conditions (c.1)-(c.3) of Sen (1967), the random variables $n^{1/2}(U^{(t)}_{n,ij} - \mu^{(t)}_{n,ij})$, $j \varepsilon S_i$, $i=1,\ldots,b$, $t=1,\ldots,p$ have jointly asymptotically a multinormal distribution with zero means and a dispersion matrix with elements $\beta^{(t,t')}_{ij,i'j'}$, defined by (5.6).

Theorem 5.1 may be readily used to derive the asymptotic normality of the joint distribution of $n^{1/2}(T^{(t)}_{N,j} - \Sigma_{i\varepsilon P_j}\mu^{(t)}_{n,ij})$, $j=1,\ldots,v$, $t=1,\ldots,p$. For the study of the asymptotic non-null distribution of $\mathcal{L}_N$, we shall consider the following sequence $\{H_n\}$ of alternative hypotheses

(5.7) H_n: $\underset{\sim}{\tau}_j = \underset{\sim}{\tau}_{j,n} = n^{-1/2}\,\underset{\sim}{\theta}_j$; $\underset{\sim}{\theta}_j = (\theta^{(1)}_j, \ldots, \theta^{(p)}_j)$, $j=1,\ldots,v$; $\Sigma\underset{\sim}{\theta}_j = \underset{\sim}{0}$, where the $\theta^{(t)}_j$ are all real and finite. For this sequence of alternatives, we shall simplify the expressions for $\mu_{n,ij}^{(t)}$ and $\beta_{ij,i'j'}^{*(t,t')}$ and then deduce the limiting distribution of the $T_{N,j}^{(t)}$. As in Sen (1967, 1968), we assume that $F_{[t]}(x)$, the marginal cdf of $e_{\alpha ij}^{(t)}$, is absolutely continuous with a continuous density function $f_{[t]}(x)$ where

(5.8) $|(d/dx)J_t[F_{[t]}(x)]|$ is bounded as $x \to \pm\infty$, for $t=1,\ldots,p$.

Let then $B_t = \int_{-\infty}^{\infty} (d/dx)J_t[F_{[t]}(x)]dF_{[t]}(x), t=1,\ldots,p$. Under (5.7) and (5.8) it follows that $F_{ij[t]}(x) = F_{[t]}(x-n^{-1/2}[\theta_j^{(t)} - \frac{1}{k}\Sigma_{\ell\varepsilon S_i}\theta_\ell^{(t)}])$ and

$$(5.9)\quad \lim_{n\to\infty} n^{1/2}(\mu_{n,ij}^{(t)} - \int_o^1 J_t(u)du) = B_t[\{\theta_j^{(t)} - \frac{1}{k}\Sigma_{\ell\varepsilon S_i}\theta_\ell^{(t)} - \tilde{\theta}_.^{(t)}] ,$$

for $j \varepsilon S_i$, $i=1,\ldots,b$, $t=1,\ldots,p$, where $\tilde{\theta}_.^{(t)} = (1/bk)\sum_{i=1}^{b}\Sigma_{j\varepsilon S_i}[\theta_j^{(t)} - \frac{1}{k}\Sigma_{\ell\varepsilon S_i}\theta_\ell^{(t)}]$, $t=1,\ldots,p$. Also,

$$(5.10)\quad \beta_{ij,i'j';rs,r's'}^{(t,t')} = 0 \text{ when } i\neq i' ,$$

and under (5.7), for all $s(s') \varepsilon S_r(S_{r'})$; $r,r'=1,\ldots,b$,

$$(5.11)\quad \lim_{n\to\infty} \beta_{ij,ij';rs,r's'}^{(t,t')} = \nu_{tt'}^{(1)}, \quad \text{if } j=j'$$

$$= \nu_{tt'}^{(2)} , \text{ if } j\neq j', i=1,\ldots,b.$$

where $\nu_{tt'}^{(i)}$ are defined by (4.1). Let then $\underset{\sim}{\xi} = (\xi_1^{(1)},\ldots,\xi_1^{(p)},\ldots,\xi_v^{(p)})$, where

$$(5.12)\quad \xi_j^{(t)} = B_t \tilde{\theta}_j^{(t)};\ \tilde{\theta}_j^{(t)} = [r_j(k-1)/k]\theta_j^{(t)} - \sum_{j'=1(\neq j)}^{v} (r_{jj'}/k)\theta_{j'}^{(t)} - r_j \tilde{\theta}_{.}^{(t)} .$$

From theorem 5.1 and the discussions following it, we obtain the following theorem by some standard computations.

Theorem 5.2. Under (5.7), (5.8) and the conditions of theorem 5.1, $[n^{1/2}(T_{N,j}^{(t)} - r_j \eta_{\cdots}^{(t)}),\ t=1,\ldots,p,\ j=1,\ldots,v]$ have (jointly) asymptotically a multinormal distribution with mean $\underset{\sim}{\xi}$ and dispersion matrix Ω, defined by (4.4).

$$\text{Let now } \underset{\sim}{\tilde{\theta}}^{o} = (\tilde{\theta}_{.}^{(1)},\ldots,\ \tilde{\theta}_{.}^{(p)},\ldots,\ \tilde{\theta}_{.}^{(1)},\ldots,\ \tilde{\theta}_{.}^{(p)}),$$

$$\underset{\sim}{\tilde{\theta}} = (\tilde{\theta}_1^{(1)},\ldots,\ \tilde{\theta}_1^{(p)},\ldots,\ \tilde{\theta}_v^{(1)},\ldots,\ \tilde{\theta}_v^{(p)})$$

and let

$$(5.13)\quad \underset{\sim}{K}_i = ((\kappa_{tt'}^{(i)}));\ \kappa_{tt'}^{(i)} = B_t \nu_{tt'}^{(i)} B_{t'}\ ,\ t,t'=1,\ldots,p;\ i=1,2,$$

where $\nu_{tt'}^{(i)}$ are defined by (4.1). Further, let

$$(5.14)\quad \underset{\sim}{\Gamma} = \underset{\sim}{A} \otimes \underset{\sim}{K}_1 - \underset{\sim}{B} \otimes \underset{\sim}{K}_2\ ,$$

where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are defined by (4.3). Finally, we denote the generalized inverse of the matrices $\underset{\sim}{\Omega}$ and $\underset{\sim}{\Gamma}$ by $\underset{\sim}{\Omega}^*$ and $\underset{\sim}{\Gamma}^*$ respectively. Now, using lemma 4.2 of Sen (1967), and proceeding along the lines of our theorem 4.1, it can be shown that under $\{H_n\}$ in (5.7), (4.5) and (4.6) still hold, and as a result, $\underset{\sim}{W}_N \sim_p \underset{\sim}{\Omega}$. As such, from (3.14) and theorem 5.2, we arrive at the following theorem by some standard analysis.

Theorem 5.3. Under the conditions of theorem 5.2, $\mathcal{L}_N$, defined by (3.14), has asymptotically a non-central chi-square distribution with p(v-1) d.f. and the non-centrality parameter

(5.15) $\Delta_{\mathcal{L}} = \underset{\sim}{\xi}\ \underset{\sim}{\Omega}^{*}\ \underset{\sim}{\xi}' = (\underset{\sim}{\tilde{\theta}} - \underset{\sim}{\tilde{\theta}}^{o})\ \underset{\sim}{\Gamma}^{*}\ (\underset{\sim}{\tilde{\theta}} - \underset{\sim}{\tilde{\theta}}^{o})'$.

REMARK. Under H_o, (5.8) is not needed, and hence, under the conditions of theorem 5.1, $\mathcal{L}_N$ has asymptotically a chi-square distribution with p(v-1) d.f.

6. Asymptotic efficiency of $\mathcal{L}_N$. In the parameter case (i.e., normal theory models), for p=1, the optimum invariant test is based on the variance-ratio criterion comparing the mean square due to treatment (eliminating blocks) with the error mean square. For $p \geq 2$, different generalizations of this test are (i) the normal theory likelihood ratio (NTLR-) test based on the likelihood criterion λ_N, (ii) the generalized (Hotelling-Lawley) T_o^2 statistic based on the trace of of $\underset{\sim}{S}_h \underset{\sim}{S}_e^{-1}$ (where $\underset{\sim}{S}_h$ and $\underset{\sim}{S}_e$ are the sum of product matrices due to the hypothesis and error, respectively,) (iii) Roy's statistic based on the largest characteristic root of $\underset{\sim}{S}_h \underset{\sim}{S}_e^{-1}$, and others. [The reader is referred to Anderson (1958, chapter 8), for details.] Unfortunately, none of these tests are uniformly (in the set of

admissible parameters) better than the others, and hence, is uniformly best. However, it follows from the results of Wald (1943) that the NTLR-test is (asymptotically) the most stringent test and and has the best average power over suitable ellipsoids in the parameter space. Also, it can be shown that the T_o^2 statistic and the NTLR-statistic are asymptotically equivalent in probability, and hence, they enjoy the same property. We shall compare the $\mathcal{L}_N$ test with the NTLR-test when the parent cdf is not necessarily normal. The distribution theory of $-2 \log\lambda_N$ when the parent distribution is not necessarily normal has been studied in the context of the general linear hypothesis by Sen and Puri (1969). It follows from their results that under H_o in (1.3), $-2 \log\lambda_N$ has asymptotically a chi-square distribution with $p(v-1)$ d.f. whenever F has finite moments up to the second order. Further, if $\underset{\sim}{e}_{\alpha ij}$, defined by (2.3) has a finite and positive-definite dispersion matrix $\underset{\sim}{\Sigma}(= \underset{\sim}{\Sigma}(F))$, it follows from theorem 2.2 of Sen and Puri (1969) and some simplifications [using some algebra essentially similar to section 6.3 of Kempthorne (1952)] that under $\{H_n\}$ in (5.7), $-2 \log\lambda_N$ has asymptotically a non-central chi-square distribution with $p(v-1)$ d.f. and the non-centrality parameter

$$(6.1) \quad \Delta_\lambda = (\underset{\sim}{\tilde{\theta}} - \underset{\sim}{\tilde{\theta}}^o) \left[\underset{\sim}{\Sigma}^{-1} \otimes \underset{\sim}{A}^{(1)*} \right] (\underset{\sim}{\tilde{\theta}} - \underset{\sim}{\tilde{\theta}}^o)' ,$$

where $\underset{\sim}{A}^{(1)*}$ is the generalized inverse of $A^{(1)}$, defined by (2.13) and (2.14). Thus, in accordance with the usual definition of the asymptotic relative efficiency (A.R.E.) (cf. [15]), the A.R.E. of the $\mathcal{L}_N$-test with respect to the λ_N-test is

$$(6.2) \quad e_{\mathcal{L},\lambda} = \Delta_{\mathcal{L}} / \Delta_{\lambda} = e(\underset{\sim}{A}^{(1)}, \underset{\sim}{A}^{(2)}, \underset{\sim}{\Sigma}, \underset{\sim}{K}_1, \underset{\sim}{K}_2, \underset{\sim}{\theta}),$$

which not only depends on (i) the design matrices $\underset{\sim}{A}^{(1)}$, $\underset{\sim}{A}^{(2)}$ and (ii) the covariance matrices $\underset{\sim}{\Sigma}$, $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$, but also on the shift $\underset{\sim}{\theta}$. For the complete block design, $\underset{\sim}{A}^{(2)}$ is a null matrix, and hence, (6.2) may be shown to be independent of $\underset{\sim}{A}^{(1)}$[cf. Sen(1968, 1969)]. We shall now prove some interesting inequalities on (6.2). For this, we let $c_m(F,\underset{\sim}{J})$ and $c_M(F,\underset{\sim}{J})$ as the minimum and maximum characteristic roots of $\underset{\sim}{\Sigma}\, \underset{\sim}{K}_1^{-1}$, where both $\underset{\sim}{\Sigma}$ and $\underset{\sim}{K}_1$ depend on F, and in addition, $\underset{\sim}{K}_1$ on the score functions $\underset{\sim}{J}$. Then, from the results of Sen and Puri (1967), it follows that $c_m(F,\underset{\sim}{J})$ and $c_M(F,\underset{\sim}{J})$ are respectively the minimum and the maximum A.R.E. of the multivariate one-sample rank order statistics (based on the score functions $\underset{\sim}{J}$) with respect to Hotelling's T^2-test. Also, let $\mathcal{D}$ be the class of all incomplete block designs D for which (2.17) holds. Then, we have the following.

<u>Theorem 6.1</u>. <u>Under the conditions of theorem</u> 5.3,

$$(6.3) \quad \inf_{\underset{\sim}{\theta}} e(\underset{\sim}{A}^{(1)}, \underset{\sim}{A}^{(2)}, \underset{\sim}{\Sigma}, \underset{\sim}{K}_1, \underset{\sim}{K}_2, \underset{\sim}{\theta}) \geq c_m(F,\underset{\sim}{J}) ,$$

(6.4) $\sup_{\underset{\sim}{\theta}} e(\underset{\sim}{A}^{(1)}, \underset{\sim}{A}^{(2)}, \underset{\sim}{\Sigma}, \underset{\sim}{K}_1, \underset{\sim}{K}_2, \underset{\sim}{\theta}) \geq c_M(F, \underset{\sim}{J})$,

<u>for all</u> $D \in \mathcal{A}$, <u>where the equality holds only when</u> $J_t[F_{[t]}(x)]$ <u>is a linear function of</u> x, <u>with probability</u> 1, <u>for all</u> $t=1,\ldots,p$. [Note that the bounds in (6.3) and (6.4) are design-free in the sense that they are independent of $\underset{\sim}{A}^{(1)}$ and $\underset{\sim}{A}^{(2)}$ for all $D \in \mathcal{A}$.]

<u>PROOF</u>. By virtue of the well-known Courant theorem on the extremum of the ratio of two non-negative definite quadratic forms, we have from (5.15), (6.1) and (6.2) that

(6.5) $\sup_{\underset{\sim}{\theta}} e_{\mathcal{L},\lambda} = c_M[\underset{\sim}{\Gamma}^*(\underset{\sim}{A}^{(1)} \otimes \underset{\sim}{\Sigma})]$ and $\inf_{\underset{\sim}{\theta}} e_{\mathcal{L},\lambda} = c_m[\underset{\sim}{\Gamma}^*(\underset{\sim}{A}^{(1)} \otimes \underset{\sim}{\Sigma})]$,

where c_m and c_M stand for the minimum and maximum characteristic roots. Now, from (4.3) and (5.14), we have

$$\begin{aligned}(6.6)\quad c_M[\underset{\sim}{\Gamma}^*(\underset{\sim}{A}^{(1)} \otimes \underset{\sim}{\Sigma})] &= 1/c_m[\underset{\sim}{\Gamma}(\underset{\sim}{\Sigma}^{-1} \otimes \underset{\sim}{A}^{(1)*})] \\ &= 1/c_m[\{\underset{\sim}{A}^{(1)} \otimes \underset{\sim}{K}_1 - \tfrac{1}{bk}(b\underset{\sim}{A}^{(1)}-(b-1)\underset{\sim}{A}^{(2)})\otimes[\underset{\sim}{K}_1+(k-1)\underset{\sim}{K}_2]\}\{\underset{\sim}{\Sigma}^{-1}\otimes\underset{\sim}{A}^{(1)*}\}] \\ &= 1/c_m[\underset{\sim}{I}_v \otimes \underset{\sim}{K}_1\underset{\sim}{\Sigma}^{-1} - \tfrac{1}{bk}[(b\underset{\sim}{A}^{(1)}-(b-1)\underset{\sim}{A}^{(2)})\underset{\sim}{A}^{(1)*}] \otimes [\underset{\sim}{K}_1+(k-1)\underset{\sim}{K}_2]\underset{\sim}{\Sigma}^{-1}] .\end{aligned}$$

Similarly,

$$\begin{aligned}(6.7)\quad c_m[\underset{\sim}{\Gamma}^*(\underset{\sim}{A}^{(1)}\otimes \underset{\sim}{\Sigma})] &= 1/c_M[\underset{\sim}{\Gamma}(\underset{\sim}{\Sigma}^{-1} \otimes \underset{\sim}{A}^{(1)*})] \\ &= 1/c_M[\underset{\sim}{I}_v \otimes \underset{\sim}{K}_1\underset{\sim}{\Sigma}^{-1} - \tfrac{1}{bk}[(b\underset{\sim}{A}^{(1)}-(b-1)\underset{\sim}{A}^{(2)})\underset{\sim}{A}^{(1)*}] \otimes [\underset{\sim}{K}_1+(k-1)\underset{\sim}{K}_2]\underset{\sim}{\Sigma}^{-1}].\end{aligned}$$

Now, by (2.17), for all $D \in \mathcal{A}$, $b\underset{\sim}{A}^{(1)}-(b-1)A^{(2)}$ is non-negative definite. Let us also prove the following Lemma.

Lemma 6.2. $\underset{\sim}{K}_1 + (k-1)\underset{\sim}{K}_2$ is positive definite unless $J_t[F_{[t]}(x)]$ is a linear function of x, with probability 1, for all $t=1,\dots,p$, and in the latter case, $\underset{\sim}{K}_1 + (k-1)\underset{\sim}{K}_2$ is a null matrix.

PROOF. By (5.13), it suffices to prove the lemma for $\underset{\sim}{\nu}^{(1)}+(k-1)\underset{\sim}{\nu}^{(2)}$. Now consider the p vector $\underset{\sim}{Z} = (Z_1,\dots,Z_p)$ where $Z_t = \Sigma_{j\varepsilon S_i} J_t[F_{[t]}(e_{\alpha ij}^{(t)})]$, $t=1,\dots,p$. Then, by definition, the dispersion matrix of $\underset{\sim}{Z}$ is

$$(6.8)\quad k[\underset{\sim}{\nu}^{(1)} + (k-1)\underset{\sim}{\nu}^{(2)}]$$

which is positive definite or null according as $\underset{\sim}{Z}$ has linearly independent elements or $\underset{\sim}{Z}$ is a constant vector, with probability 1. Since $\Sigma_{j\varepsilon S_i} e_{\alpha ij}^{(t)} = 0$, with probability one, for $t=1,\dots,p$, the rest of the arguments follow as in lemma 4.5 of Sen (1968). Q. E. D.

Thus, from (6.6), (6.7) and lemma 6.2, we have for all $D \varepsilon \mathfrak{D}$,

$(1/bk)[(b\underset{\sim}{A}^{(1)}-(b-1)\underset{\sim}{A}^{(2)})\underset{\sim}{A}^{(1)*}]\otimes[(\underset{\sim}{K}_1+(k-1)\underset{\sim}{K}_2)]\underset{\sim}{\Sigma}^{-1}$ is non-negative definite.

Hence

$$(6.9)\quad c_M[\underset{\sim}{\Gamma}*(\underset{\sim}{A}^{(1)}\otimes \underset{\sim}{\Sigma})] \geq 1/c_m[\underset{\sim}{I}_v \otimes \underset{\sim}{K}_1\underset{\sim}{\Sigma}^{-1}] = c_M(\underset{\sim}{\Sigma}\underset{\sim}{K}_1^{-1}) ,$$

$$(6.10)\quad c_m[\underset{\sim}{\Gamma}*(\underset{\sim}{A}^{(1)} \otimes \underset{\sim}{\Sigma})] \geq 1/c_M[\underset{\sim}{I}_v \otimes \underset{\sim}{K}_1\underset{\sim}{\Sigma}^{-1}] = c_m(\underset{\sim}{\Sigma}\ \underset{\sim}{K}_1^{-1}) ,$$

where the equality signs hold if and only if $J_t[F_{[t]}(x)]$ is a linear function of x, with probability one, for all $t=1,\dots,p$. The rest of the proof follows directly upon noting that

$$(6.11)\quad c_m(F,\underset{\sim}{J}) = c_m[\underset{\sim}{\Sigma}\ \underset{\sim}{K}_1^{-1}] \text{ and } c_M(F,\underset{\sim}{J}) = c_M[\underset{\sim}{\Sigma}\ \underset{\sim}{K}_1^{-1}].$$ Q. E. D.

In the particular case of $p = 1$,

(6.12) $c_m[\underset{\sim}{\Sigma}\ \underset{\sim}{K}_1^{-1}] = c_M[\underset{\sim}{\Sigma}\ \underset{\sim}{K}_1^{-1}] = B_1^2\ \sigma_{11}/\nu_{11}^{(1)}$,

where B_1 and $\nu_{11}^{(1)}$ are defined just after (5.8) and in (4.1), respectively. Now, noting that the right hand side of (6.12) agrees with (4.14) of Sen (1968), we arrive at the following.

Theorem 6.3. The ARE results and bound studied in section 4 of Sen (1968) remain valid for the entire class $\mathcal{D}$ of incomplete block designs D.

Thus, if for J_t, we use the normal scores, the A.R.E. of the corresponding $\mathcal{L}_N$-test with respect to the variance-ratio test is bounded below by 1, for all $F_{[1]}$. The lower bound is attained only for normal $F_{[1]}$.

For $p \geq 2$, by virtue of theorem 6.2, the A.R.E. results of Sen and Puri (1967) [for the multivariate one sample location problem] supply the lower bounds for the desired A.R.E. results. When F is itself multinormal, and we use the normal scores, it follows that $\underset{\sim}{K}_1 + (k-1)\underset{\sim}{K}_2 = \underset{\sim}{0}^{p\times p}$, and hence, by theorem 6.2 and the results of Sen and Puri (1967), it follows that the normal scores $\mathcal{L}_N$ test and the λ_N-test are asymptotically power equivalent. For the bounds for $c_m(F,\underset{\sim}{J})$ and $c_M(F,\underset{\sim}{J})$ in some other special cases, the reader is referred to Sen and Puri (1967), for details.

In the particular case of p=1, it is of some interest to compare our tests with the ones available in the literature (cf. [2], [3], [4], [6], [7]); for $p > 1$, no other test is proposed as yet. The A.R.E. of the median procedures with respect to the classical ANOVA-test has been studied by Bhapkar (1961, 1963), and it follows from his results that this A.R.E. is in general, quite low, particularly for blocks of small number of plots. The A.R.E. of the rank-sum test by Durbin (1951) has been studied by Elteren and Noether (1959), only for the balanced incomplete block (BIB) design. Arguing in the same way as in Elteren and Noether (1959) and generalizing their results to the general class of incomplete block designs, considered in this paper, it can be shown that the A.R.E. of the Benard - Elteren test with respect to the classical ANOVA-test is

$$(6.13) \quad [12\, \sigma^2(1-\rho_\varepsilon)k/(k+1)]\ [\int_{-\infty}^{\infty} g^*(x,x)\,dx]^2$$

where $\sigma^2 = \mathrm{Var}(\varepsilon_{\alpha ij})$, $\rho_\varepsilon\sigma^2 = \mathrm{Covar}(\varepsilon_{\alpha ij}, \varepsilon_{\alpha ij'})$, $j,j' \in S_i$ and $g^*(x,y)$ is the joint density function of $(\varepsilon_{\alpha ij}, \varepsilon_{\alpha ij'})$, $j,j' \in S_i$, whose marginal densities are both equal to $g(x)$. [If the errors in the block are independent, $\rho_\varepsilon = 0$ and $g^*(x,y) = g(x)g(y)$.]

Let us now compare the A.R.E. of the $\mathcal{L}_N$ test with that in (6.13) in the particular case when $\mathcal{L}_N$ is based on aligned rank-sums (i.e., $J(n) = u : o < u < 1$). Using the results already derived in this section, it can be easily shown that the A.R.E. of the Benard-Elteren test with respect to the $\mathcal{L}_N$-test (when J(w = u) is <u>bounded above</u> by

$$(6.14) \quad [k^2/(k^2-1)] \left[\int_{-\infty}^{\infty} g^*(x,x)dx \right]^2 / \left[\int_{-\infty}^{\infty} f^2(x)dx \right]^2 ,$$

where f(x) is the marginal density of $e_{\alpha ij}$. When G is a k-variate normal cdf, (6.15) reduces to

$$(6.15) \quad k/(k+1) \; (< 1),$$

which is independent of ρ_ε. Thus, the proposed aligned rank sum test is more efficient for normal alternatives.

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