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ON A CLASS OF RANK ORDER TESTS FOR REGRESSION WITH PARTIALLY INFORMED STOCHASTIC PREDICTORS

by

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0. Summary. Hájek (1962) has obtained asymptotically most powerful rank order tests for simple (linear) regression with non-stochastic predictors. The present paper extends the findings of Hájek to the multiple (linear) regression model with stochastic predictors including the situations where the predictors are partially informed. The proposed tests are shown to be permutationally (conditionally) distribution-free. Their asymptotic properties and efficiencies are studied and the asymptotic optimality is established under the conditions of Wald (1943).

1. Introduction. Consider a (double) sequence of stochastic matrices $\underset{\sim}{Z}_\nu=(\underset{\sim}{Z}_{\nu 1},\ldots,\underset{\sim}{Z}_{\nu\nu})$, $(1 \leq \nu < \infty)$, where $\underset{\sim}{Z}'_{\nu i} = (Y_{\nu i}, \underset{\sim}{X}'_{\nu i}) = (Y_{\nu i}, X^{(1)}_{\nu i},\ldots, X^{(p)}_{\nu i})$, $(p \geq 1)$ are independent and identically distributed random vectors (iidrv) having an absolutely continuous distribution function (d.f.) $F_\nu(y, \underset{\sim}{x})$, $\underset{\sim}{x} \in R^p$, the p-dimensional real space. We assume that the p-variate marginal d.f. $F_{01}(\underset{\sim}{x}) = F_\nu(\infty, \underset{\sim}{x})$ does not depend on ν and possesses a continuous density function $f_{01}(\underset{\sim}{x})$. We denote the conditional density function of $Y_{\nu i}$ given $\underset{\sim}{X}_{\nu i} = \underset{\sim}{x}$ by $f_\nu(y|\underset{\sim}{x})$. We also assume that $f_\nu(y|\underset{\sim}{x}) = f_{20}([y - \alpha - \nu^{-\frac{1}{2}}\underset{\sim}{\beta}'\underset{\sim}{x}]/\sigma)$, where α and σ (>0) are nuisance parameters and $\beta_1,\ldots, \beta_p$ are parameters under test. Thus the density function corresponding to the d.f. $F_\nu(y, \underset{\sim}{x})$ is given by

$$(1.1)\quad f_\nu(y, \underset{\sim}{x}) = f_{01}(\underset{\sim}{x})\, f_{20}([y - \alpha - \nu^{-1/2}\, \underset{\sim}{\beta}'\, \underset{\sim}{x}]/\sigma),\ \nu \geq 1.$$

We want to test the null hypothesis of no regression i.e.

$$(1.2)\quad H_o : \underset{\sim}{\beta} = \underset{\sim}{0} \text{ against the alternatives } \underset{\sim}{\beta} \neq \underset{\sim}{0}.$$

It may be noted that under $H_o : \underset{\sim}{\beta} = \underset{\sim}{0}$, $f_\nu(y, \underset{\sim}{x}) = f(y, \underset{\sim}{x}) = f_{01}(\underset{\sim}{x})\, f_{20}([y-\alpha]/\sigma)$, for all $\nu \geq 1$ and thus $Y_{\nu i}$ and $\underset{\sim}{X}_{\nu i}$ are stochastically independent for all

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$i=1,2,\ldots,\nu$ and all $\nu \geq 1$. We shall use the notations $\overline{X}_\nu^{(k)} = \nu^{-1} \sum_{i=1}^{\nu} X_{\nu i}^{(k)}$ $(k = 1,\ldots,p)$ and $\overline{Y}_\nu = \nu^{-1} \sum_{i=1}^{\nu} Y_{\nu i}$.

The following assumptions are made:

(1.3) $f'_{20}(y) = (d/dy)\, f_{20}(y)$ exists and $I(f_{20}) = \int_{-\infty}^{\infty} (f'_{20}/f_{20})^2\, f_{20} dx < \infty$;

(1.4) $E(X_{\nu 1}^{(k)}) = 0$ (assumed without any loss of generality) and $E|X_{\nu 1}^{(k)}|^{4+\delta} < c < \infty$, for some $\delta > 0$ and for all $k=1,\ldots,p$, $\nu \geq 1$;

(1.5) $\underset{\sim}{\Sigma} = \mathrm{Var}(\underset{\sim}{X}_{\nu 1})$ is positive definite (p.d.)

Let

(1.6) $\phi(u) = - f'_{20}\,(F_{20}^{-1}(u))/f_{20}(F_{20}^{-1}(u))$, $0 < u < 1$,

where $F_{20}^{-1}(u) = \inf\,\{x : F(x) \geq u\}$;

(1.7) $\phi^*(u) = - g'\,(G^{-1}(u))/g(G^{-1}(u))$, $0 < u < 1$,

where G is a class of d.f. with density function g satisfying (1.3); let

(1.8) $I(g) = \int_0^1 \phi^{*2}(u)\, du$.

We introduce a class of score functions $a_\nu(i)$, $i=1,2,\ldots,\nu$, satisfying

(1.9) $\lim_{\nu \to \infty} \int_0^1 \,[a_\nu(1 + [u\nu]) - \phi^*(u)]^2\, du = 0$,

where [x] denotes the highest integer less than or equal to x. In particular, we can take the following types of score functions as suggested by Hâjek [12]:

(a) $a_\nu(i) = E[\phi^*\,(U_\nu^{(i)})]$, $i=1,\ldots,\nu$, $\nu \geq 1$;

(b) $a_\nu(i) = \phi^*(i/(\nu + 1))$, $i=1,\ldots,\nu$, $\nu \geq 1$;

(c) $a_\nu(i) = \nu \int_{(i-1)/\nu}^{i/\nu} \phi^*(u)\, du$, $i=1,\ldots,\nu$, $\nu \geq 1$, where $U_{\nu 1}^{(1)} \leq \ldots \leq U_\nu^{(\nu)}$ denote ordered statistics in a sample $U_{\nu 1},\ldots, U_{\nu\nu}$ of size ν from a uniform distribution on (0, 1), $U_{\nu i}$'s being distributed independently of $X_{\nu i}^{(k)}$'s $(i=1,\ldots,\nu;\ k=1,\ldots,p)$. Let $F_k(x)$ be the marginal distribution of $X_{\nu 1}^{(k)}$ and and $F_{k,k'}(x, y)$ the joint bivariate marginal distribution of $X_{\nu 1}^{(k)}$ and $X_{\nu 1}^{(k')}$.

We consider another class of score functions denoted by $b_{k\nu}(i)$, $(i=1,\ldots,\nu, \nu \geq 1;\ k=1,\ldots,p)$ satisfying

$$\lim_{\nu \to \infty} \int_0^1 [b_{k\nu}(1 + [u\nu]) - \psi_k^*(u)]^2 \, du = 0, \tag{1.10}$$

where $\psi_k^*(u) = F_k^{*-1}(u)$, $(k=1,\ldots,p)$, F_k^* denoting the d.f. of a random variable (r.v.) $W_\nu^{(k)}$, $(k=1,\ldots,p)$, satisfying (1.4) and (1.5). We also assume that $(W_\nu^{(1)},\ldots,W_\nu^{(p)})$ are distributed independently of $Y_{\nu 1}, \ldots, Y_{\nu\nu}$.

In Section 2, a class of permutationally (conditionally) invariant distribution-free tests have been proposed and studied. In Section 3, the asymptotic permutation distribution of the proposed class of test statistics (under H_o) has been developed. The asymptotic non-null distributions have been obtained in Section 4 by using the contiguity lemmas of LeCam (See e.g. [11] or [12]). In Section 5, the optimality properties of the proposed tests have been established under the conditions of Wald [17] and the ARE results of the allied tests (in Pitman's sense) have been derived.

In the remainder of this section, we prove a lemma which will be used repeatedly in the subsequent sections.

LEMMA 1.1. Let $\{X_{\nu i};\ i=1,2,\ldots,\nu, \nu \geq 1\}$ be a double sequence of random variables such that for each ν, $X_{\nu 1},\ldots, X_{\nu\nu}$ are independently distributed with d.f. $F_\nu(x)$ for which $E|X_{\nu i}|^{2+\delta} < c < \infty$ for some $\delta > 0$ and all $i=1,\ldots,\nu$. Then, $\bar{X}_\nu - E(\bar{X}_\nu) \to 0$ almost everywhere (a.e.)

The proof of the above lemma is based on the following result :

LEMMA 1.2. Let $X_{\nu 1}, X_{\nu 2}, X_{\nu 3},\ldots; X_{\nu\nu}$ be samples from distributions with d.f. $F_{\nu 1}(x), F_{\nu 2}(x),\ldots, F_{\nu\nu}(x)$ respectively. If there exists $\eta (> 2)$ such that $E|X_{\nu i}|^\eta \leq c < \infty$, for all $i=1,\ldots,\nu$, $\nu \geq 1$, then given any $\varepsilon > 0$,

$$P\{|\bar{X}_\nu - \bar{\mu}_\nu| > \varepsilon\} < K/(\varepsilon^\eta \nu^{\eta/2}) \quad \text{for all } \nu > \nu_0 \text{ and some positive } K, \text{ where}$$

$$\bar{X}_\nu = \nu^{-1} \sum_{i=1}^{\nu} X_{\nu i} \text{ and } \mu_\nu = \nu^{-1} \sum_{i=1}^{\nu} E(X_{\nu i}).$$

PROOF. The proof of the lemma follows exactly in the same lines as that of Brillinger [5] (who however has considered the case of iidrv $X_1, X_2,\ldots$ with $E|X_1|^\eta < \infty$),

$\nu!$ permutations of the columns of $\underset{\sim}{R}^*_{\nu(0)}$ would be uniform under H_o, i.e.

(2.4) $$P\{\underset{\sim}{R}_{\nu(0)} = \underset{\sim}{r}_{\nu(0)} \,|\, \Sigma(\underset{\sim}{R}^*_{\nu(0)}),\ H_o\} = (\nu!)^{-1} \text{ for all } \underset{\sim}{r}_{\nu(0)} \varepsilon\ \Sigma(\underset{\sim}{R}^*_{\nu(0)})$$

irrespective of $F_{01}(\underset{\sim}{x})$. Let $\mathcal{P}_\nu$ denote the permutational (conditional) probability measure generated by the conditional law in (2.4). Then, after some algebraic computations, we get the following results:

(2.5) $$E(S_{k\nu} \,|\, \mathcal{P}_\nu) = 0, \ k=1,\dots,p;$$

(2.6) $$V_{kk',\nu} = \text{Cov}\,(S_{k\nu}, S_{k'\nu} \,|\, \mathcal{P}_\nu)$$

$$= (\nu/(\nu-1))\{\nu^{-1} \sum_{i=1}^{\nu} (b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu})\ (b_{k'\nu}(R_{\nu i}^{(k')}) - \bar{b}_{k'\nu})\} \ .$$

$$\{\nu^{-1} \sum_{i=1}^{\nu} (a_\nu(i) - a_\nu)^2\}\ , \ (k,\ k'=1,\dots,p).$$

We propose the following test statistic:

(2.7) $$M_\nu = \underset{\sim}{S}'_\nu\ \underset{\sim}{V}^*_\nu\ \underset{\sim}{S}_\nu\ ,$$

where $\underset{\sim}{V}^*_\nu$ is a generalized inverse of $\underset{\sim}{V}_\nu = ((V_{kk'\nu}))$. We shall show in the next section that under the assumptions (1.3), (1.4), (1.9) and (1.10), $\underset{\sim}{V}_\nu$ converges in probability to the matrix $\underset{\sim}{\Sigma}_0\ I(g)$, where

(2.8) $$\underset{\sim}{\Sigma}_0 = ((\sigma^0_{kk'}))$$

with $\sigma_{kk} = \int_0^1 \psi_k^{*2}(u)\,du$, $k=1,\dots,p$, and

$$\sigma_{kk'} = \int_0^1 \int_0^1 \psi_k^*(u)\ \psi_{k'}^*(u')\ dH_{kk'}(u,u'),\ 1 \le k \ne k' \le p\ ,$$

$H_{k,k'}(u,u')$ being the bivariate joint d.f. of $F_k(X_{\nu 1}^{(k)})$ and $F_{k'}(X_{\nu 1}^{(k')})$, $1 \le k \ne k' \le p$.
The following test procedure is proposed based on the critical function $\zeta_1(Z_\nu)$:

(2.9) $$\zeta_1(\underset{\sim}{Z}_\nu) = \begin{cases} 1 & \text{if } M_\nu > M_{\nu,\varepsilon} \\ \delta_{1\nu\varepsilon} & \text{if } M_\nu = M_{\nu,\varepsilon} \\ 0 & \text{if } M_\nu < M_{\nu,\varepsilon} \end{cases} ,$$

where $M_{\nu,\varepsilon}$ and $\delta_{1\nu\varepsilon}$ are so chosen that $E_{\mathcal{P}_\nu}(\zeta_1(\underset{\sim}{Z}_\nu)) = \varepsilon$. This implies that $E(\zeta_1(\underset{\sim}{Z}_\nu)|H_o) = \varepsilon$, $E(\cdot|H_o)$ standing for expectation under H_o i.e. $\zeta_1(\underset{\sim}{Z}_\nu)$ is a similar size ε test. However, this procedure requires evaluation of $\nu!$ possible realizations of M_ν . The task becomes prohibitively laborious for large ν and this leads us to the study of the asymptotic distribution of the permutation test statistic M_ν .

3. Asymptotic permutation distribution of M_ν. We shall now prove a theorem which will be used in deriving the asymptotic permutation distribution of M_ν. This theorem generalizes Theorem 4.2 of Puri and Sen (1966) in the sense that unlike Puri and Sen we do not have to assume the existence of the derivatives of the score functions $\phi^*(u)$ or $\psi_k^*(u)$'s and the boundedness conditions thereon. In proving the result all we need to assume is the square integrability of ϕ^* and the uniform boundedness of the $(2+\delta)^{\text{th}}$ order moments of $W_{\nu i}^{(k)}$'s $(i=1,\ldots,\nu,\ k=1,\ldots,p)$ for some $\delta > 0$; the latter implies the conditions $|\psi_k^*(u)| \leq [u(1-u)]^{-1/2+\delta'}$ for some appropriate $\delta' > 0$ as assumed in [16].

THEOREM 3.1. $\underset{\sim}{V}_\nu \to I(g)\, \underset{\sim}{\Sigma}_0$ in probability

PROOF. $\nu^{-1} \sum_{i=1}^{\nu} [a_\nu(i) - \bar{a}_\nu]^2 = \nu^{-1} \sum_{i=1}^{\nu} a_\nu^2(i) - \bar{a}_\nu^2$.

Now, $\bar{a}_\nu = \int_0^1 a_\nu(1+[u\nu])du = \int_0^1 [a_\nu(1+[u\nu]) - \phi^*(u)]du$ (since $\int_0^1 \phi^*(u)du=0$).

Using now (1.9) and the Schwarz inequality, we get, $\bar{a}_\nu \to 0$ as $\nu \to \infty$. Further,

$$\nu^{-1} \sum_{i=1}^{\nu} a_\nu^2(i) = \int_0^1 a_\nu^2(1+[u\nu])du$$

$$= \int_0^1 [a_\nu(1+[u\nu]) - \phi^*(u)]^2 du + 2\int_0^1 \phi^*(u)\, [a_\nu(1+[u\nu]) - \phi^*(u)]\, du + \int_0^1 \phi^{*2}(u)du$$

Using now, (1.9) and the fact that

$$\int_0^1 |\phi^*(u)\{a_\nu(1+[u\nu]) - \phi^*(u)\}|\, du \leq I^{1/2}(g)\, \{\int_0^1 [a_\nu(1+[u\nu]) - \phi^*(u)]^2 du\}^{1/2}$$

with $I(g) < \infty$, we get, $\nu^{-1} \sum_{i=1}^{\nu} a_\nu^2(i) \to I(g)$. Thus,

$$\nu^{-1} \sum_{i=1}^{\nu} [a_\nu(i) - \bar{a}_\nu]^2 \to I(g). \tag{3.1}$$

Proceeding exactly in the same way and using (1.10), we can show that,

$$\nu^{-1} \sum_{i=1}^{\nu} [b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu}]^2 = \nu^{-1} \sum_{i=1}^{\nu} [b_{k\nu}(i) - \bar{b}_{k\nu}]^2 \to \sigma_{kk}^0 \quad (k=1,\ldots,p). \tag{3.2}$$

Using (2.6), (3.1) and (3.2) we get,

(3.3) $$V_{kk,\nu} \to \sigma^0_{kk}\, I(g) \quad \text{as } \nu \to \infty .$$

For $k \neq k'$, we get from (2.6),

$$V_{kk'} = (\nu/(\nu-1))\{\nu^{-1} \sum_{i=1}^{\nu} b_{k\nu}(R^{(k)}_{\nu i})\; b_{k'\nu}(R^{(k')}_{\nu i}) - \bar{b}_{k\nu}\,\bar{b}_{k'\nu}\} .$$

$$\{\nu^{-1} \sum_{i=1}^{\nu} a^2_\nu(i) - \bar{a}^2_\nu\} .$$

For proving the lemma, all we have to show now is that

(3.4) $$\nu^{-1} \sum_{i=1}^{\nu} b_{k\nu}(R^{(k)}_{\nu i})\; b_{k'\nu}(R^{(k')}_{\nu i}) \to \sigma^0_{kk'}, \text{ in probability.}$$

We introduce the following notations.

(3.5) $$H_{kk'\nu}(u,v) = \nu^{-1}[\#\text{ of } \{F_k(X^{(k)}_{\nu i}),\, F_{k'}(X^{(k')}_{\nu i})\} \leq (u,v)] ;$$

(3.6) $$H_{k\nu}(u) = \nu^{-1}\, [\#\text{ of } F_k\, (X^{(k)}_{\nu i}) \leq u] ;$$

(3.7) $$C_{k\nu}(i/(\nu+1)) = b_{k\nu}(i); \quad (i=1,\ldots,\nu;\ 1 \leq k \neq k' \leq p).$$

Hence,

(3.8) $$\nu^{-1} \sum_{i=1}^{\nu} b_{k\nu}(R^{(k)}_{\nu i}) b_{k'\nu}(R^{(k')}_{\nu i}) = \int_0^1\!\!\int_0^1 C_{k\nu}(\nu H_{k\nu}(u)/(\nu+1)) C_{k'\nu}(\nu H_{k'\nu}(v)/(\nu+1))\, dH_{kk'\nu}(u,v)$$

$$= \int_0^1\!\!\int_0^1 [C_{k\nu}(\nu H_{k\nu}(u)/(\nu+1)) - \psi^*_k(\nu H_{k\nu}(u)/(\nu+1))] C_{k'\nu}(\nu H_{k'\nu}(v)/(\nu+1))\, dH_{kk'\nu}(u,v)$$

$$+ \int_0^1\!\!\int_0^1 [C_{k'\nu}(\nu H_{k'\nu}(v)/(\nu+1)) - \psi^*_{k'}(\nu H_{k'\nu}(v)/(\nu+1))]\; \psi^*_k(\nu H_k(u)/(\nu+1))\, dH_{kk'}(u,v)$$

$$+ \int_0^1\!\!\int_0^1 [\psi^*_k(\nu H_{k\nu}(u)/(\nu+1)) - \psi^*_k(u)]\; \psi^*_{k'}(\nu H_{k'\nu}(v)/(\nu+1))\, dH_{kk'\nu}(u,v)$$

$$+ \int_0^1\!\!\int_0^1 [\psi^*_{k'}(\nu H_{k'\nu}(v)/(\nu+1)] - \psi^*_{k'}(v)]\; \psi^*_k(u)\, dH_{kk'\nu}(u,v)$$

$$+ \int_0^1\!\!\int_0^1 \psi^*_k(u)\; \psi^*_{k'}(v)\, dH_{kk'}\,(u,v)$$

$$= (I) + (II) + (III) + (IV) + (V) , \text{ (say)}.$$

$$|(I)| \leq \{\int_0^1 [C_{k\nu}(\nu H_{k\nu}(u)/(\nu+1)) - \psi^*_k\, (\nu H_{k\nu}(u)/(\nu+1))]^2 dH_{k\nu}(u)\}^{1/2} .$$

$$\{\int_0^1 C^2_{k'\nu}(\nu H_{k'\nu}(v)/(\nu+1))\, dH_{k'\nu}(v)\}^{1/2}$$

$$= \{\, \nu^{-1} \sum_{i=1}^{\nu} [C_{k\nu}(R^{(k)}_{\nu i}/(\nu+1)) - \psi^*_k(R^{(k)}_{\nu i}/(\nu+1))]^2\}^{1/2} .$$

$$\{\nu^{-1}\sum_{i=1}^{\nu} C^2_{k',\nu}(R^{(k')}_{\nu i}/(\nu+1))\}^{1/2}$$

$$= \{\nu^{-1}\sum_{i=1}^{\nu} [b_{k\nu}(i) - \psi^*_k (i/(\nu+1))]^2\}^{1/2} \{\nu^{-1}\sum_{i=1}^{\nu} b^2_{k\nu}(i)\}^{1/2}$$

Now, $\nu^{-1}\sum_{i=1}^{\nu} [b_{k\nu}(i) - \psi^*_k(i/(\nu+1))]^2 = \int_0^1 [b_{k\nu}(1+[u\nu]) - \psi^*_k((1+[u\nu])/(\nu+1))]^2 du$

$\leq 2 \int_0^1 [b_{k\nu}(1+[u\nu]) - \psi^*_k(u)]^2 du + 2 \int_0^1 [\psi^*_k(1+[u\nu]/(\nu+1)) - \psi^*_k(u)]^2 du$

Since $\psi^*_k(u)$ is a monotonically non-decreasing and square integrable function of u, using lemma a (p. 164) of [12], we get,

$\int_0^1 [\psi^*_k ((1+[u\nu])/(\nu+1)) - \psi^*_k(u)]^2 du \to 0$ as $\nu \to \infty$. Using now (1.10), we get from (3.9), (I) $\to 0$ as $\nu \to \infty$. Similarly (II) $\to 0$ as $\nu \to \infty$.

(3.10) $$|(III)| \leq [\int_0^1 \{\psi^*_k(\nu H_{k\nu}(u)/(\nu+1)) - \psi^*_k(u)\}^2 dH_{k\nu}(u)]^{1/2} \cdot [\int_0^1 \psi^{*2}_{k'} (\nu H_{k',\nu}(v)/(\nu+1)) dH_{k',\nu}(v)]^{1/2}$$

But,

(3.11) $$\int_0^1 \psi^{*2}_{k'} (\nu H_{k',\nu}(v)/(\nu+1))\, dH_{k',\nu}(v) \to \sigma^0_{k'k'} \text{ as } \nu \to \infty.$$

Also,

(3.12) $$E \int_0^1 [\psi^*_k (\nu H_{k\nu}(u)/(\nu+1)) - \psi^*_k(u)]^2 dH_{k\nu}(u)$$

$$= E[\nu^{-1}\sum_{i=1}^{\nu} \{\psi^*_k(R^{(k)}_{\nu i}/(\nu+1)) - \psi^*_k(U_{\nu i})\}^2]$$

$$= E[\nu^{-1}\sum_{i=1}^{\nu} (\{ \psi^{**}_k (R^{(k)}_{\nu i}/(\nu+1)) - \psi^{**}_k(U_{\nu i})\} + \{\psi^{**}_k(U_{\nu i}) - \psi^*_k(U_{\nu i})\}^2]$$

where $\psi^{**}_k(\lambda)$ is a function defined on $0 < \lambda \leq 1$ by

(3.13) $$\psi^{**}_k(\lambda) = \psi^*_k (i/(\nu+1)) \text{ for } (i-1)/\nu < \lambda \leq i/\nu , (i=1,\ldots,\nu)$$

so that $\psi^{**}_k(i/(\nu+1)) = \psi^*_k(i/(\nu+1))$ for all $i=1,\ldots,\nu$. Then using the elementary inequality $(a+b)^2 \leq 2(a^2 + b^2)$, we get from (3.12),

(3.14) $$E \int_0^1 [\psi^*_k(\nu H_{k\nu}(u)/(\nu+1)) - \psi^*_k(u)]^2 dH_{k\nu}(u)$$

$$\leq 2\, E [\nu^{-1}\sum_{i=1}^{\nu} \{ \psi^{**}_k(R^{(k)}_{\nu i}/(\nu+1)) - \psi^{**}_k(U_{\nu i})\}^2]$$

$$+ 2\ E\ [\nu^{-1} \sum_{i=1}^{\nu} \{\psi_k^{**}(U_{\nu i}) - \psi_k^{*}(U_{\nu i})\}^{2}].$$

$$= 2\ E\ [\psi_k^{**}\ (R_{\nu 1}^{(k)}/(\nu+1)) - \psi_k^{**}(U_{\nu 1})]^2$$

$$+ 2 \int_0^1 [\psi_k^{**}(u) - \psi_k^{*}(u)]^2 du$$

Now, $\int_0^1 [\psi_k^{**}(u) - \psi_k^{*}(u)]^2 du = \int_0^1 [\psi_k^{*}(1+[u\nu]/(\nu+1)) - \psi_k^{*}(u)]^2\ du$

Since $\psi_k^{*}(u)$ is a monotonically non-decreasing and square integrable function of u, using lemma a, p. 164 of [12], $\int_0^1 [\psi_k^{**}(u) - \psi_k^{*}(u)]^2 du \to 0$ as $\nu \to \infty$. Again using lemma 2.1 of [12], we get,

(3.15) $E\ [\ \psi_k^{**}\ (R_{\nu 1}^{(k)}/(\nu+1)) - \psi_k^{**}(U_{\nu 1})]^2$

$$\leq 2\sqrt{2}\ (\nu^{-1/2} \max_{1\leq i\leq \nu} |\ \psi_k^{**}(i/(\nu+1)) - \overline{\psi}_k^{**}|)(\nu^{-1} \sum_{i=1}^{\nu} (\psi_k^{**}(i/(\nu+1)) - \overline{\psi}_k^{**})^2)^{1/2},$$

where $\overline{\psi}_k^{**} = \nu^{-1} \sum_{i=1}^{\nu} \psi_k^{**}(i/(\nu+1)) = \nu^{-1} \sum_{i=1}^{\nu} \psi_k^{*}(i/(\nu+1)) \to \int_0^1 \psi_k^{*}\ (u) du = 0$

as $\nu \to \infty$. Also, $\nu^{-1} \sum_{i=1}^{\nu} \psi^{**2}\ (i/(\nu+1)) \to \sigma_{kk}^0 < \infty$, and

$$\nu^{-1/2} \max_{1\leq i\leq \nu} |\psi_k^{**}(i/(\nu+1))| = \nu^{1/2} \max_{1\leq i\leq \nu} |\int_{(i-1)/\nu}^{i/\nu} \psi_k^{*}(u) du| \leq \max_{1\leq i\leq \nu} \{\int_{(i-1)/\nu}^{i/\nu} \psi_k^{*2}(u) du\}^{1/2} \to 0$$

as $\nu \to \infty$ since $\sigma_{kk}^0 < \infty$. Now, from (3.15), $E\ [\ \psi_k^{**}(R_{\nu 1}^{(k)}/(\nu+1)) - \psi_k^{**}(U_{\nu 1})]^2 \to 0$ as $\nu \to \infty$. Thus, from (3.14),

(3.16) $E \int_0^1 [\psi_k^{*}\ (\nu H_{k\nu}(u)/(\nu+1)) - \psi_k^{*}(u)]^2\ dH_{k\nu}(u) \to 0$ as $\nu \to \infty$.

Using (3.11) and (3.16) with the fact that convergence in mean implies convergence in probability we get now from (3.10), (III) $\to 0$ in probability as $\nu \to \infty$. Similarly (IV) $\to 0$ in probability as $\nu \to \infty$. Again,

(V) $= \nu^{-1} \sum_{i=1}^{\nu} \psi_k^{*}\ (F_k(X_{\nu i}^{(k)}))\ \psi_{k'}^{*}(F_{k'}(X_{\nu i}^{(k')}))$. Since,

$$E\ |\ \psi_k^{*}\ (F_k(X_{\nu i}^{(k)}))\ \psi_{k'}^{*}(F_{k'}(X_{\nu i}^{(k')}))\ |^{1+\frac{\delta}{2}}$$

$\le \{E|\psi_k^*(F_k(X_{\nu i}^{(k)}))|^{2+\delta} \; E|\psi_{k'}^*(F_{k'}(X_{\nu i}^{(k')}))|^{2+\delta}\}^{1/2} < \infty$, using Markov's weak law of large numbers, we get,

$$(V) \to \nu^{-1} \sum_{i=1}^{\nu} E[\psi_k^*(F_k(X_{\nu i}^{(k)}))\, \psi_{k'}^*(F_{k'}(X_{\nu i}^{(k')}))] = \sigma_{kk'}^0, \quad \text{in probability as } \nu \to \infty.$$

Hence (3.4) is proved.

The above theorem is used in proving the following theorem:

THEOREM 3.2. Under the permutational model $\mathcal{P}_\nu$, $\underset{\sim}{S}_\nu$ is asymptotically p-variate normal $(\underset{\sim}{0}, I(g)\cdot\underset{\sim}{\Sigma}_0)$ in probability.[1]

PROOF. Let $\underset{\sim}{e} = (e_1,\ldots,e_p)' \ne \underset{\sim}{0}$ be a vector of finite real constants. Then

$$(3.17) \quad E(\underset{\sim}{e}'\underset{\sim}{S}_\nu|\mathcal{P}_\nu) = 0,\; E[(\underset{\sim}{e}'\underset{\sim}{S}_\nu)^2|\mathcal{P}_\nu] = (\underset{\sim}{e}'\underset{\sim}{V}_\nu\underset{\sim}{e})I(g).$$

We have to show that for every vector $\underset{\sim}{e}$ as above, $\underset{\sim}{e}'\underset{\sim}{S}_\nu$ is under $\mathcal{P}_\nu$ asymptotically normal $(0,(\underset{\sim}{e}'\underset{\sim}{\Sigma}_0\underset{\sim}{e})I(g))$ in probability. We have,

$$(3.18) \quad \underset{\sim}{e}'\underset{\sim}{S}_\nu = \sum_{i=1}^{\nu} \{\sum_{k=1}^{p} e_k(b_k(R_{\nu i}^{(k)}) - \bar{b}_{k\nu})\}\{a_\nu(R_{\nu i}^{(0)}) - \bar{a}_\nu\}.$$

Under $\mathcal{P}_\nu$, $\{\sum_{k=1}^{p} e_k(b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu}),\ i=1,\ldots,\nu\}$ is equiprobable over the $\nu!$ permutations of the columns of $R_{\nu(0)}^*$ while $\{a_\nu(i),\ i=1,\ldots,\nu\}$ is equiprobable over the $\nu!$ permutations of the integers $1,2,\ldots,\nu$ independently of the permutations of $\{\sum_{k=1}^{p} e_k(b_{k\nu}(R_{\nu i}^{(k)}))\ i=1,\ldots,\nu\}$. Since $\underset{\sim}{V}_\nu \to \underset{\sim}{\Sigma}_0$ in probability, to prove the theorem it is sufficient to show that the sequence $\{\sum_{k=1}^{p} e_k b_k(R_{\nu i}^{(k)})\}$ satisfies the Noether condition (see e.g. [10]) in probability and the sequence $\{a_\nu(i)\}$ satisfies the condition

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[1]By saying $\underset{\sim}{X}$ is asymptotically p-variate normal $(\underset{\sim}{\mu},\underset{\sim}{\Sigma})$, where $\underset{\sim}{\Sigma}$ is positive definite, we mean for every vector $\underset{\sim}{a} = (a_1,\ldots,a_p)' \ne \underset{\sim}{0}$ of finite real constants $(\underset{\sim}{a}'\underset{\sim}{x} - \underset{\sim}{a}'\underset{\sim}{\mu})/(\underset{\sim}{a}'\underset{\sim}{\Sigma}\underset{\sim}{a})^{1/2}$ has asymptotically a normal (0, 1) distribution.

(3.19) $$\lim_{\nu\to\infty} k_\nu/\nu = 0 \Rightarrow \lim_{\nu\to\infty} \frac{\max\limits_{1\le i_1\le \dots \le i_{k\nu}\le\nu} \sum_{\alpha=1}^{k_\nu} [a_\nu(i_\alpha) - \bar{a}_\nu]^2}{\sum_{i=1}^{\nu} [a_\nu(i) - \bar{a}_\nu]^2} = 0 ,$$

which implies the Noether condition,
and then apply Theorem 4.2. of [12], p. 514. Let $a^2_{\nu(1)} \le a^2_{\nu(2)} \le \dots \le a^2_{\nu(\nu)}$ be the ordered $a^2_\nu(i)$'s and let us define a step function $\eta_\nu(u)$ as

(3.20) $\eta_\nu(u) = a^2_\nu(i)$ for $(i-1)/\nu < u \le i/\nu$, $i=1,\dots,\nu$.

Then $\nu^{-1} \max\limits_{1\le i_1<\dots i_{k_\nu}<\nu} \sum_{\alpha=1}^{k_\nu} [a^2_\nu(i_\alpha)] = \int_{1-k_\nu/\nu}^{1} \eta_\nu(u)du$. But,

$\int_0^1 \eta_\nu(u) = \nu^{-1} \sum_{i=1}^{\nu} a^2_\nu(i) \to I(g) < \infty$ as $\nu \to \infty$. Hence, $\int_{1-k_\nu/\nu}^{1} \eta_\nu(u)du \to 0$ as $\nu \to \infty$ since then $k_\nu/\nu \to 0$. Also $\bar{a}_\nu \to 0$ as $\nu \to \infty$. Hence,

(3.21) $$\frac{\max\limits_{1\le i_1<\dots<i_{k_\nu}<\nu} \sum_{\alpha=1}^{k_\nu} [a_\nu(i_\alpha) - \bar{a}_\nu]^2}{\sum_{i=1}^{\nu} [a_\nu(i) - \bar{a}_\nu]^2} \le \frac{2\,\nu^{-1} \max\limits_{1\le i_1<\dots<i_{k_\nu}\le\nu} \sum_{\alpha=1}^{k_\nu} a^2_\nu(i_\alpha) + 2\bar{a}^2_\nu}{\nu^{-1} \sum_{i=1}^{\nu} a^2_\nu(i) - \bar{a}^2_\nu}$$

$\to 0$ as $\nu \to \infty$.

Putting in particular $k_\nu = 1$, it follows that the sequence $\{a_\nu(i)\}$ satisfies the Noether condition i.e.

(3.22) $$\frac{\max\limits_{1\le i\le\nu} [a_\nu(i) - \bar{a}_\nu]^2}{\sum_{i=1}^{\nu} [a_\nu(i) - \bar{a}_\nu]^2} \to 0 \text{ as } \nu \to \infty .$$

Also,

(3.23) $$\nu^{-1} \max_{1\le i\le\nu} \{ \sum_{k=1}^{p} e_k [b_{k\nu}(R^{(k)}_{\nu i}) - \bar{b}_{k\nu}]\}^2$$

$$\le \nu^{-1} \max_{1\le i\le\nu} p \sum_{k=1}^{p} e^2_k [b_{k\nu}(R^{(k)}_{\nu i}) - \bar{b}_{k\nu}]^2$$

$$\leq \ (p \sum_{k=1}^{p} e_k^2) \max_{1 \leq i \leq \nu} \ \max_{1 \leq k \leq p} \nu^{-1}[b_{k\nu}(i) - \bar{b}_{k\nu}]^2 \ \to 0 \text{ as } \nu \to \infty$$

(the proof being similar as in the case of $a_\nu(i)$'s.). Further,

$$(3.24) \quad \nu^{-1} \sum_{i=1}^{\nu} \{ \sum_{k=1}^{p} e_k [b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu}]\}^2 = \underset{\sim}{e}' \underset{\sim\nu}{V} \underset{\sim}{e} \to \underset{\sim}{e}' \underset{\sim 0}{\Sigma} \underset{\sim}{e} > 0 \text{ in probability}$$

(using lemma 3.1 and the fact that $\underset{\sim 0}{\Sigma}$ is p.d.). Hence,

$$(3.25) \quad \frac{\max_{1 \leq i \leq \nu} \{ \sum_{k=1}^{p} e_k [b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu}]\}^2}{\sum_{i=1}^{\nu} \{ \sum_{k=1}^{p} e_k [b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu}]\}^2} \to 0 \quad \text{in probability.}$$

Hence the theorem. An immediate consequence of the above theorem is

THEOREM 3.3. <u>Under</u> $\mathcal{P}_\nu$, M_ν <u>is asymptotically a central</u> χ^2 <u>with</u> p <u>degrees of freedom</u>.

4. <u>Asymptotic non-null distribution of</u> M_ν. We first prove the following elementary lemma:

LEMMA 4.1. (i) $\nu^{-1/2} \max_{1 \leq i \leq \nu} |X_{\nu i}^{(k)}| \to 0$ <u>a.e. as</u> $\nu \to \infty$ <u>for all</u> $k=1,\ldots,p$.

(ii) $\nu^{-1} \sum_{i=1}^{\nu} (X_{\nu i}^{(k)} - \bar{X}_\nu^{(k)})^2 \to E\,(X_{\nu i}^{(k)})^2$ <u>as</u> $\nu \to \infty$, <u>for all</u> $k=1,\ldots,p$.

PROOF. We first prove (ii). We can write,

$$(4.1) \quad \nu^{-1} \sum_{i=1}^{\nu} (X_{\nu i}^{(k)} - \bar{X}_\nu^{(k)})^2 = \nu^{-1} \sum_{i=1}^{\nu} (X_{\nu i}^{(k)})^2 - \bar{X}_\nu^{(k)2} .$$

Since $(X_{\nu 1}^{(k)})^2, \ldots, (X_{\nu\nu}^{(k)})^2$ are i.i.d. with $E|(X_{\nu 1}^{(k)})^2|^{2+\frac{\delta}{2}} = E|X_{\nu i}^{(k)}|^{4+\delta} < c < \infty$, using lemma 1.1,

$$(4.2) \quad E\,[\nu^{-1} \sum_{i=1}^{\nu} (X_{\nu i}^{(k)})^2] \to E\,(X_{\nu 1}^{(k)})^2 \quad \text{a.e.}$$

Again, $E|X_{\nu 1}^{(k)}|^{2+\delta} \leq (E|X_{\nu 1}^{(k)}|^{4+\delta})^{\frac{2+\delta}{4+\delta}} < c^{\frac{2+\delta}{4+\delta}} < \infty$. So,

$$(4.3) \quad \bar{X}_\nu^{(k)} \to E\,(X_{\nu 1}^{(k)}) = 0 \quad \text{a.e.} \quad \text{as } \nu \to \infty.$$

(ii) follows now from (4.1) - (4.3).

To prove (i), we first recall two equivalent forms of Noether condition given by Hoeffding (1951). We state these in a slightly different form convenient for our purpose:

$$(4.4)\ \left(\max_{1 \le i \le \nu} (X_{\nu i}^{(k)} - \bar{X}_{\nu}^{(k)})^2\right) / \left(\sum_{i=1}^{\nu} (X_{\nu i}^{(k)} - \bar{X}_{\nu}^{(k)})^2\right) \to 0 \text{ a.e. as } \nu \to \infty$$

$$\Longleftrightarrow \lim_{\nu \to \infty} \left(\sum_{i=1}^{\nu} |X_{\nu i}^{(k)} - \bar{X}_{\nu}^{(k)}|^r\right) / \left[\sum_{i=1}^{\nu} (X_{\nu i}^{(k)} - \bar{X}_{\nu}^{(k)})^2\right]^{r/2} = 0 \text{ a.e., for some } r>2.$$

The right hand expression in (4.4.) can be rewritten as

$$(4.5)\quad \lim_{\nu \to \infty} \nu^{1-\frac{r}{2}} \{\nu^{-1} \sum_{i=1}^{\nu} |X_{\nu i}^{(k)} - \bar{X}_{\nu}^{(k)}|^r\} / \{\nu^{-1} \sum_{i=1}^{\nu} (X_{\nu i}^{(k)} - \bar{X}_{\nu}^{(k)})^2\}^{r/2} = 0 \text{ a.e. for some } r>2.$$

We shall prove first (4.5). Taking $r = 2 + \delta'$ where $(0 < \delta' < \delta/2)$ and using an elementary inequality, we get

$$(4.6)\quad \nu^{-1} \sum_{i=1}^{\nu} |X_{\nu i}^{(k)} - \bar{X}_{\nu}^{(k)}|^{2+\delta'} \le 2^{1+\delta'} (\nu^{-1} \sum_{i=1}^{\nu} |X_{\nu i}^{(k)}|^{2+\delta'} + |\bar{X}_{\nu}^{(k)}|^{2+\delta'})$$

Taking now, $\delta'' = \dfrac{\delta - 2\delta'}{2 + \delta'}$, we have,

$$\left(E|X_{\nu i}^{(k)}|^{2+\delta'}\right)^{2+\delta''} \le E\,|X_{\nu i}^{(k)}|^{(2+\delta')(2+\delta'')} = E|X_{\nu i}^{(k)}|^{4+\delta} < c < \infty.$$ Hence, by lemma 1.1,

$$(4.7)\qquad \nu^{-1} \sum_{i=1}^{\nu} |X_{\nu i}^{(k)}|^{2+\delta'} \to E|X_{\nu 1}^{(k)}|^{2+\delta'} \text{ a.e. as } \nu \to \infty .$$

Also $\bar{X}_{\nu}^{(k)} \to 0$ a.e. Using now (ii), (4.6) and (4.7), we get (4.5). (i) now follows from (4.4), (4.5), (4.7) and the fact $\bar{X}_{\nu}^{(k)} \to 0$ a.e. as $\nu \to \infty$. Q. E. D.

Let P_ν and Q_ν be the probability distributions of $\underset{\sim}{Z}_\nu$ under the null hypothesis and alternatives

$$(4.8)\quad \beta_k = \beta_{k0}\ ,\ \sigma = \sigma_0\ ,\ (k=1,\ldots,p).$$

Let $P_{\nu i}$ and $Q_{\nu i}$ be the respective d.f. of $\underset{\sim}{Z}_{\nu i}$ under H_o and H_ν $(i=1,\ldots,\nu;\ \nu \ge 1)$. Let $p_{\nu i}(\cdot)$ and $q_{\nu i}(\cdot)$ be the corresponding densities. Define

$$(4.9)\quad r_{\nu i}(x) = \begin{cases} q_{\nu i}(x)/p_{\nu i}(x), & \text{if } p_{\nu i}(x) > 0 \\ 0, & \text{otherwise} \end{cases} ;\ (i=1,\ldots,\nu).$$

Consider now the following two statistics:

$$(4.10)\quad W_\nu = 2 \sum_{i=1}^{\nu} (r_{\nu i}^{1/2} - 1),\quad L_\nu = \prod_{i=1}^{\nu} r_{\nu i} .$$

We also introduce the notations :

(4.11) $\gamma_k = \beta_{k0}/\sigma_0$, $(k=1,\ldots,p)$; $\underset{\sim}{\beta}_0' = (\beta_{10},\ldots,\beta_{p0})$; $\underset{\sim}{\gamma}' = (\gamma_1,\ldots,\gamma_p)$.

The asymptotic distribution of M_ν will be obtained by using the contiguity lemmas of LeCam. We need show that the distributions Q_ν are contiguous to the distributions P_ν. We adopt a procedure analogous to that of Hájek (1962). The following lemma will be needed:

LEMMA 4.2. $E(W_\nu|P_\nu) \sim -\frac{1}{4} I(f_{20}) (\underset{\sim}{\gamma}' \underset{\sim}{\Sigma} \underset{\sim}{\gamma})^1$

PROOF. $W_\nu = 2\sum_{i=1}^{\nu}(r_{\nu i}^{1/2}-1) = 2\sum_{i=1}^{\nu}[\{f_{20}(\sigma^{-1}(Y_{\nu i}-\alpha-\nu^{-1/2}\underset{\sim}{\beta}_0'\underset{\sim}{X}_{\nu i}))/f_{20}(\sigma^{-1}(Y_{\nu i}-\alpha))\}^{\frac{1}{2}}-1]$

Further writing $(Y_{\nu i}-\alpha)/\sigma = Y'_{\nu i}$, $h_{\nu i} = \nu^{-1/2}\underset{\sim}{\gamma}'\underset{\sim}{X}_{\nu i}$ $(i=1,\ldots,\nu)$ and $s_{20}(x) = f_{20}^{1/2}(x)$, we get,

(4.12) $W_\nu = 2\sum_{i=1}^{\nu}[s_{20}(Y'_{\nu i}-h_{\nu i})/s_{20}(Y'_{\nu i})-1]$.

But, $\max_{1\le i\le\nu}|h_{\nu i}| \le (\max_{1\le k\le p}\max_{1\le i\le\nu}|X_{\nu i}^{(k)}|/\nu^{1/2})(\sum_{k=1}^{p}|\gamma_k|) \to 0$ a.e. as $\nu\to\infty$ (using lemma 4.1(i)). Also, $E(h_{\nu i}^2|P_\nu) = \underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\gamma}$ which is non-zero and finite. We get from (4.12) after some algebraic simplifications,

$$E(W_\nu|P_\nu) = -\sum_{i=1}^{\nu} E(h_{\nu i}^2\, g(h_{\nu i})),$$

where $g(h_{\nu i}) = \int_{-\infty}^{\infty}[\{s_{20}(y-h_{\nu i})-s_{20}(y)\}/h_{\nu i}]^2\,dy \to \frac{1}{4} I(f_{20})$ a.e. Proceeding now exactly as in ([12], lemma a, p. 211), we get the result. Q. E. D.

Let,

(4.13) $$T_k = \nu^{-1/2}\sum_{i=1}^{\nu}X_{\nu i}^{(k)}\phi(F_{20}(Y'_{\nu i})) = \nu^{-1/2}\sum_{i=1}^{\nu}X_{\nu i}^{(k)}(-f'_{20}(Y'_{\nu i})/f_{20}(Y'_{\nu i}))$$
$$= -2\nu^{-1/2}\sum_{i=1}^{\nu}X_{\nu i}^{(k)}[s'_{20}(Y'_{\nu i})/s_{20}(Y'_{\nu i})].$$

Proceeding as in Hájek [12], we get the following lemma:

LEMMA 4.3. $\mathrm{Var}(W_\nu - \underset{\sim}{\gamma}'\underset{\sim}{T}_\nu \mid P_\nu) \to 0$ as $\nu\to\infty$, where $\underset{\sim}{T}_\nu = (T_{1\nu},\ldots,T_{p\nu})'$.

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[1] By $a_\nu \sim b_\nu (b_\nu \neq 0)$, we mean $a_\nu/b_\nu \to 1$ as $\nu\to\infty$.

Using lemmas 4.2 and 4.3, it now follows that

$$E\ [\ (W_\nu - \underset{\sim}{\gamma}'\ \underset{\sim}{T}_\nu + \frac{1}{4} I(f_{20})\ \underset{\sim}{\gamma}'\ \underset{\sim}{\Sigma}\ \underset{\sim}{\gamma})^2\ |\ P_\nu] \to 0 \qquad \text{which implies}$$

(4.14) $W_\nu - \underset{\sim}{\gamma}'\ \underset{\sim}{T}_\nu + \frac{1}{4} I(f_{20})\ \underset{\sim}{\gamma}'\ \underset{\sim}{\Sigma}\ \underset{\sim}{\gamma} \to 0$ in P_ν-probability.

LEMMA 4.4. <u>Under</u> P_ν, T_ν <u>is asymptotically multinormal</u> $(\underset{\sim}{0}, I(f_{20})\ \underset{\sim}{\Sigma})$ <u>a.e</u>.

PROOF. We have to show that for every vector $\underset{\sim}{e} = (e_1,\dots,e_p)' \neq 0$ of finite real constants, $\underset{\sim}{e}'\underset{\sim}{T}_\nu$ is asymptotically normal $(0, I(f_{20})\ \underset{\sim}{e}'\ \underset{\sim}{\Sigma}\ \underset{\sim}{e})$. We can write,

(4.15) $$\underset{\sim}{e}'\underset{\sim}{T}_\nu = \nu^{-1/2} \sum_{i=1}^{\nu} \left(\sum_{k=1}^{p} e_k X_{\nu i}^{(k)}\right) \phi(F_{20}(Y_{\nu i}')).$$

(4.16) $$E\ [\ \phi(F_{20}(Y_{\nu i}'))\,|P_\nu] = 0, \quad E[\phi^2(F_{20}(Y_{\nu i}'))\,|P_\nu] = I\ (f_{20}) < \infty.$$

Also, $\max_{1 \le i \le \nu} \left| \sum_{k=1}^{p} e_k X_{\nu i}^{(k)} / \nu^{1/2} \right| \to 0$ as $\nu \to \infty$ from lemma 4.1 (i) and

$\sum_{i=1}^{\nu} \{ \sum_{k=1}^{p} e_k X_{\nu i}^{(k)} \}^2 \to \underset{\sim}{e}'\ \underset{\sim}{\Sigma}\ \underset{\sim}{e}$ a.e. as $\nu \to \infty$ from lemma 4.1 (ii) . So,

$\max_{1 \le i \le \nu} \left(\sum_{k=1}^{p} e_k X_{\nu i}^{(k)} \right)^2 / \sum_{i=1}^{\nu} \{ \sum_{k=1}^{p} e_k X_{\nu i}^{(k)} \}^2 \to 0$ a.e. as $\nu \to \infty$. i.e.

(4.17) the sequence $\{ \sum_{k=1}^{p} e_k X_{\nu i}^{(k)} \}$ satisfies the Noether condition a.e.

Now, under P_ν, $Y_{\nu i}'$'s are independent of $\sum_{k=1}^{p} e_k X_{\nu i}^{(k)}$'s and so are $\phi(F_{20}(Y_{\nu i}'))$'s which are Borel-measurable functions of $Y_{\nu i}'$'s. Since (4.16) and (4.17) are satisfied and $(F_{20}(Y_{\nu 1}')),\dots,$ $(F_{20}(Y_{\nu\nu}'))$ are all identically distributed, conditional on the $X_{\nu i}^{(k)}$'s fixed $\underset{\sim}{e}'\ \underset{\sim}{T}_\nu$ is asymptotically normal

$(0, (\underset{\sim}{\gamma}'\ ((\nu^{-1} \sum_{i=1}^{\nu} X_{\nu i}^{(k)} X_{\nu i}^{(k')}))\ \underset{\sim}{\gamma}) I(f_{20}))$. Since $((\nu^{-1} \sum_{i=1}^{\nu} X_{\nu i}^{(k)} X_{\nu i}^{(k')})) \to \underset{\sim}{\Sigma}$ a.e., we get, $\underset{\sim}{e}'\ \underset{\sim}{T}_\nu$ is unconditionally asymptotically normal $(0, (\underset{\sim}{e}'\ \underset{\sim}{\Sigma}\ \underset{\sim}{e}) I(f_{20}))$a.e. Q. E. D.

Using the above lemma, we get now from (4.14),

(4.18) W_ν is asymptotically normal $(-\frac{1}{4} I(f_{20})\ \underset{\sim}{\gamma}'\ \underset{\sim}{\Sigma}\ \underset{\sim}{\gamma},\ I(f_{20})\ \underset{\sim}{\gamma}'\ \underset{\sim}{\Sigma}\ \underset{\sim}{\gamma})$.

We next prove the following lemma.

LEMMA 4.5. $\lim_{\nu\to\infty} \max_{1\le i\le\nu} P_\nu(|q_{\nu i}(\tilde{Z}_{\nu i})/p_{\nu i}(\tilde{Z}_{\nu i}) - 1| > \varepsilon) = 0$

PROOF. $P_\nu(|q_{\nu i}(\tilde{Z}_{\nu i})/p_{\nu i}(\tilde{Z}_{\nu i}) - 1| > \varepsilon)$

$$\le \varepsilon^{-1} E[\,|q_{\nu i}(\tilde{Z}_{\nu i})/p_{\nu i}(\tilde{Z}_{\nu i}) - 1|\,|P_\nu]$$

$$= \varepsilon^{-1} E[\,|h_{\nu i}| \int_{-\infty}^{\infty} |\{f_{20}(y-h_{\nu i}) - f_{20}(y)\}/h_{\nu i}|\, dy$$

But it can be shown after some elementary algebra that

$$\int_{-\infty}^{\infty} |\{f_{20}(y-h_{\nu i}) - f_{20}(y)\}/h_{\nu i}|\,dy \le I^{1/2}(f_{20}), \quad (i=1,\ldots,\nu)$$

Hence, $P_\nu(|q_{\nu i}(\tilde{Z}_{\nu i})/p_{\nu i}(\tilde{Z}_{\nu i}) - 1| > \varepsilon)$

$$\le \varepsilon^{-1} I^{1/2}(f_{20})\, E|h_{\nu i}|. \quad \text{So,}$$

(4.19) $\max_{1\le i\le\nu} P_\nu(|q_{\nu i}(\tilde{Z}_{\nu i})/p_{\nu i}(\tilde{Z}_{\nu i}) - 1| > \varepsilon)$

$$\le \varepsilon^{-1} I^{1/2}(f_{20}) \max_{1\le i\le\nu} E|h_{\nu i}|$$

$$\le \varepsilon^{-1} I^{1/2}(f_{20})\, E(\max_{1\le i\le\nu} |h_{\nu i}|).$$

We know, $\max_{1\le i\le\nu} |h_{\nu i}| \to 0$ a.e. and

(4.20) $$\max_{1\le i\le\nu} |h_{\nu i}| \le \nu^{-1/2}\{\sum_{k=1}^{p} \max_{1\le i\le\nu} |X_{\nu i}^{(k)}|\}\,(\max_{1\le k\le p}|\gamma_k|).$$

But, $\int_{A_k} \nu^{-1/2} \max_{1\le i\le\nu} |X_{\nu i}^{(k)}(w)|\,dP_{0k}(w) = \int_{A_k} \nu^{-1/2}(\max_{1\le i\le\nu}|X_{\nu i}^{(k)}(w)|)I_{A_k}(w)\,dP_{0k}^{(w)}$

(where $A_k \in \mathcal{A}_k$, $(\Omega_k, \mathcal{A}_k, P_{0k})$ being the measure space and

$$I_{A_k}(w) = \begin{cases} 1 & \text{if } w \in A_k \\ 0 & \text{if } w \in \Omega_k - A_k) \end{cases}$$

$$\le [\int \nu^{-1}\{\max_{1\le i\le\nu}|X_{\nu i}^{(k)}(w)|\}^2 dP_{0k}(w)]^{1/2}[\int I_{A_k}(w)\,dP_{0k}(w)]^{1/2}$$

$$= \{\nu^{-1} E[\max_{1\le i\le\nu}|X_{\nu i}^{(k)}|]^2\}^{1/2} \{P_{0k}(A_k)\}^{1/2}$$

$\le \{\nu^{-1}(\nu^2/(2\nu-1))\,\sigma_k^2\}^{1/2} P_{0k}^{1/2}(A_k) \le \sigma_k P_{0k}^{1/2}(A_k) \to 0$ uniformly in ν as

$P_{0k}(A_k) \to 0$. Hence, integrals of r.v. $\nu^{-1/2}\max_{1\le i\le\nu}|X_{\nu i}^{(k)}(w)|$ are uniformly continuous for each k=1,...,p and using (4.20) and the L_r-convergence theorem of Loéve ([15], p.163), with r=1, we get,

$$\text{(4.21)}\quad E(\max_{1 \le i \le \nu} |h_{\nu i}|) \to 0 \quad \text{as } \nu \to \infty .$$

The theorem now follows from (4.19).

We now state LeCam's second lemma (see[12], p.205) which is an immediate consequence of (4.12) and lemma 4.5.

LEMMA 4.6. _The statistic_ $\log L_\nu$ _satisfies_

(i) $\log L_\nu - W_\nu + \frac{1}{4} I(f_{20})(\underset{\sim}{\gamma}' \underset{\sim}{\Sigma}\, \underset{\sim}{\gamma}) \to 0$ _in_ P_ν_-probability_.

(ii) $\log L_\nu$ _is asymptotically normal_ $(-\frac{1}{2} I(f_{20})(\underset{\sim}{\gamma}' \underset{\sim}{\Sigma}\, \underset{\sim}{\gamma}), I(f_{20})(\underset{\sim}{\gamma}' \underset{\sim}{\Sigma}\, \underset{\sim}{\gamma}))$ _a.e._

(iii) _the distributions_ Q_ν _are contiguous to the distributions_ P_ν .

We are now in a position to derive the asymptotic non-null distribution of $\underset{\sim}{S}_\nu$ by using LeCam's third lemma (see [12], p. 208). We first define the following statistics:

$$\text{(4.22)}\quad S^*_{k\nu} = \nu^{-1/2} \sum_{i=1}^{\nu} [b_{k\nu}(R^{(k)}_{\nu i}) - \bar{b}_{k\nu}]\, \phi^*(F_{20}(Y'_{\nu i})),\ k=1,\dots,p;$$

$$\text{(4.23)}\quad T^*_{k\nu} = \nu^{-1/2} \sum_{i=1}^{\nu} \psi^*_k(U_{\nu i})\, \phi^*(F_{20}(Y'_{\nu i})),\ k=1,\dots,p.$$

Then,

$$\text{(4.24)}\quad E[(S^*_{k\nu} - T^*_{k\nu})^2 | P_\nu] = E[(\nu^{-1/2} \sum_{i=1}^{\nu} \{b_{k\nu}(R^{(k)}_{\nu i}) - \bar{b}_{k\nu} - \psi^*_k(U_{\nu i})\}\, \phi^*_k(U_{\nu i}))^2 \mid P_\nu\}$$

$$= \nu^{-1} \sum_{i=1}^{\nu} E[b_{k\nu}(R^{(k)}_{\nu i}) - \bar{b}_{k\nu} - \psi^*_k(U_{\nu i})]^2\, I(g)$$

$$\le [2\, \nu^{-1} \sum_{i=1}^{\nu} E[b_{k\nu}(R^{(k)}_{\nu i}) - \psi^*_k(U_{\nu i})]^2 + 2\, \bar{b}^2_{k\nu}]\, I(g)$$

But $I(g) < \infty$, $\bar{b}_{k\nu} \to 0$ as $\nu \to \infty$ and proceeding as in Theorems 1.5. a (p. 157) and 1.6. a (p. 163) of [12] $E[b_{k\nu}(R^{(k)}_{\nu i}) - \psi^*_k(U_{\nu i})]^2 \to 0$ as $\nu \to \infty$, for all $i=1,\dots,\nu$. Hence, from (4.24),

$$\text{(4.25)}\quad E[(S^*_{k\nu} - T^*_{k\nu})^2 \mid P_\nu] \to 0 \quad \text{as } \nu \to \infty, \text{ for all } k=1,\dots,p.$$

Hence, $S^*_{k\nu} - T^*_{k\nu} \to 0$ in P_ν-probability (and hence in Q_ν-probability because of contiguity) for all k=1,...,p.

Using Bonferrouni inequality, we get,

(4.26) $\underset{\sim}{S}_\nu^* - \underset{\sim}{T}_\nu^* \to \underset{\sim}{0}$ in P_ν-probability (and hence Q_ν-probability),

where $\underset{\sim}{S}_\nu^* = (S_{1\nu}^*, \ldots, S_{p\nu}^*)'$ and $\underset{\sim}{T}_\nu^* = (T_{1\nu}^*, \ldots, T_{p\nu}^*)'$. Further,

$$S_{k\nu} - S_{k\nu}^* = \nu^{-1/2} \sum_{i=1}^{\nu} [b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu}]\,[a_\nu(R_{\nu i}^{(0)}) - \bar{a}_\nu],\ k=1,\ldots,p.$$

Hence,

(4.27) $E[(S_{k\nu} - S_{k\nu}^*)^2 \mid P_\nu]$

$$= E[\nu^{-1} \sum_{i=1}^{\nu} (b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu})^2 (a_\nu(R_{\nu i}^{(0)}) - \phi^*(U_{\nu i}))^2$$

$$+ \nu^{-1} \sum_{i \neq i'} (b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu})(b_{k\nu}(R_{\nu i'}^{(k)}) - \bar{b}_{k\nu}) \cdot$$

$$(a_\nu(R_{\nu i}^{(0)}) - \phi^*(U_{\nu i}))(a_\nu(R_{\nu i'}^{(0)}) - \phi^*(U_{\nu i'}))|P_\nu]$$

$$= \nu^{-1} \sum_{i=1}^{\nu} E(b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu})^2\, E(a_\nu(R_{\nu 1}^{(0)}) - \phi^*(U_{\nu 1}))^2$$

$$+ \nu^{-1} \sum_{i \neq i'} E\{(b_{k\nu}(R_{\nu i}^{(k)}) - \bar{b}_{k\nu})\ (b_{k\nu}(R_{\nu i'}^{(k)}) - \bar{b}_{k'\nu})\} \cdot$$

$$E[(a_\nu(R_{\nu 1}^{(0)}) - \phi^*(U_{\nu 1}))\ (a_\nu(R_{\nu 2}^{(0)}) - \phi^*(U_{\nu 2}))]$$

$$\leq \nu^{-1} \sum_{i=1}^{\nu} (b_{k\nu}(i) - \bar{b}_{k\nu})^2\ \{E[a_\nu(R_{\nu 1}^{(0)}) - \phi^*(U_{\nu 1})]^2 + (E[a_\nu(R_{\nu 1}^{(0)}) - \phi^*(U_{\nu 1})]^2 \cdot$$

$$E[a_\nu(R_{\nu 2}^{(0)}) - \phi^*(U_{\nu 2})]^2)^{1/2}\}.$$

But,

$$\nu^{-1} \Sigma[b_{k\nu}(i) - \bar{b}_{k\nu}]^2 \to \sigma_{kk}^0 < \infty \text{ and } E[a_\nu(R_{\nu i}^{(0)}) - \phi^*(U_{\nu i})]^2 \to 0 \text{ for all } i=1,\ldots,\nu.$$

Hence, from (4.27),

(4.28) $E[(S_{k\nu} - S_{k\nu}^*)^2 \mid P_\nu] \to 0$ as $\nu \to \infty$ for all $k=1,\ldots,p$.

Hence,

(4.29) $\underset{\sim}{S}_\nu - \underset{\sim}{S}_\nu^* \to \underset{\sim}{0}$ in P_ν (and hence Q_ν)- probability.

(4.26) and (4.29) give,

(4.30) $\underset{\sim}{S}_\nu - \underset{\sim}{T}_\nu^* \to 0$ in P_ν (and hence Q_ν)- probability.

From (4.14) and lemma 4.6(i), $\log L_\nu - \underset{\sim}{\gamma}'\, \underset{\sim}{T}_\nu + \frac{1}{2} I(f_{20})\, \underset{\sim}{\gamma}'\, \underset{\sim}{\Sigma}\, \underset{\sim}{\gamma} \to 0$ in

P_ν-probability (also used in proving lemma 4.6(ii)). Hence, for any vector $\underset{\sim}{e} \neq \underset{\sim}{0}$ of finite real constants, $(\log L_\nu, \underset{\sim}{e}'\underset{\sim}{S}_\nu)$ has asymptotically the same distribution as $(\underset{\sim}{\gamma}' \underset{\sim}{T}_\nu - \frac{1}{2} I(f_{20}) \underset{\sim}{\gamma}' \underset{\sim}{\Sigma} \underset{\sim}{\gamma}, \underset{\sim}{e}'\underset{\sim}{S}_\nu)$. Using (4.30) the latter has the same asymptotic distribution as $(\underset{\sim}{\gamma}' \underset{\sim}{T}_\nu - \frac{1}{2} I(f_{20}) \underset{\sim}{\gamma}' \underset{\sim}{\Sigma} \underset{\sim}{\gamma}, \underset{\sim}{e}' \underset{\sim}{T}^*_\nu)$. Let

(4.31) $\underset{\sim}{\Sigma}_{00} = \int_0^1 \int_0^1 \psi^*_k(u)\psi_{k'}(u')dH_{kk'}(u,u'),\ \psi_k(u) = F_k^{-1}(u),\ k,k'=1,\ldots,p.$

We prove the following lemma:

LEMMA 4.7. <u>Under</u> P_ν, $(\underset{\sim}{\gamma}'\underset{\sim}{T}_\nu - \frac{1}{2} I(f_{20})\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\gamma}, \underset{\sim}{e}'\underset{\sim}{T}^*_\nu)$ <u>is asymptotically bivariate normal</u> $(-\frac{1}{2} I(f_{20})\underset{\sim}{\gamma}' \underset{\sim}{\Sigma} \underset{\sim}{\gamma}, 0, I(f_{20})\underset{\sim}{\gamma}' \underset{\sim}{\Sigma} \underset{\sim}{\gamma}), (\underset{\sim}{e}' \underset{\sim}{\Sigma}_0 \underset{\sim}{e})I(g), (\underset{\sim}{e}' \underset{\sim}{\Sigma}_{00}\underset{\sim}{\gamma})\int_0^1 \phi(u)\phi^*(u)du)$.

PROOF. We have to show that for every $(a,b) \neq (0,0)$ where a,b are finite real constants, $a\, \underset{\sim}{\gamma}'\underset{\sim}{T}_\nu + b\underset{\sim}{e}'\underset{\sim}{T}^*_\nu$ is asymptotically normal $(0,\ a^2(\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\, \underset{\sim}{\gamma})I(f_{20}) + b^2(\underset{\sim}{e}' \underset{\sim}{\Sigma}_0\, \underset{\sim}{e})I(g) + 2ab(\underset{\sim}{e}' \underset{\sim}{\Sigma}_{00}\, \underset{\sim}{\gamma}) \int_0^1 \phi(u)\, \phi^*(u)du)$. Now,

$$a\, \underset{\sim}{\gamma}'\underset{\sim}{T}_\nu + b\underset{\sim}{e}'\underset{\sim}{T}^*_\nu = \nu^{-1/2} \sum_{i=1}^{\nu} [\sum_{k=1}^{p} \{a\gamma_k\psi_k(U_{\nu i}) + be_k\psi^*_k(U_{\nu i})\}]\phi^*(F_{20}(Y'_{\nu i}))$$

$$= \sum_{i=1}^{\nu} Z^*_{\nu i} \text{ (say)},$$

where $Z^*_{\nu i} = Z^*_{1\nu i} + Z^*_{2\nu i}$ $(i=1,\ldots,\nu)$, $Z^*_{1\nu i} = \nu^{-1/2} \sum_{k=1}^{p} a\gamma_k\psi_k(U_{\nu i})$, $Z^*_{2\nu i} = \nu^{-1/2} \sum_{k=1}^{p} be_k\psi^*_k(U_{\nu i})$, $i=1,\ldots,\nu$. We may also note that $E[a\underset{\sim}{\gamma}'\underset{\sim}{T}_\nu + b\underset{\sim}{e}'\underset{\sim}{T}^*_\nu|H_o] = 0$ and $\mathrm{Var}(a\underset{\sim}{\gamma}'\underset{\sim}{T}_\nu + b\underset{\sim}{e}'\underset{\sim}{T}^*_\nu|H_o) = a^2(\underset{\sim}{\gamma}'\underset{\sim}{\Sigma}\, \underset{\sim}{\gamma})I(f_{20}) + b^2(\underset{\sim}{e}'\underset{\sim}{\Sigma}_0\underset{\sim}{e})\, I(g) + 2ab\, (\underset{\sim}{e}'\underset{\sim}{\Sigma}_{00}\, \underset{\sim}{\gamma}) \int_0^1 \phi(u)\, \phi^*(u)du$. As in [12] (pp. 217-218) all we need to show now is that $\sum_{i=1}^{\nu} E[Z^{*2}_{\nu i}\, \chi_{Z^*_{\nu i}}(\delta)|P_\nu] \to 0$ as $\nu \to \infty$, where for any $\delta > 0$,

$$\chi_{Z^*_{\nu i}}(\delta) = \begin{cases} 1 & \text{if } |Z^*_{\nu i}| > \delta \\ 0 & \text{otherwise}. \end{cases}$$

But, $\chi_{Z^*_{\nu i}}(\delta) \leq \chi_{Z^*_{1\nu i}}(\delta/2) + \chi_{Z^*_{2\nu i}}(\delta/2)$. Hence, $\sum_{i=1}^{\nu} E[Z^{*2}_{\nu i}\, \chi_{Z^*_{\nu i}}(\delta)|P_\nu]$

$$\leq 2 \sum_{i=1}^{\nu} E[(Z^{*2}_{1\nu i} + Z^{*2}_{2\nu i})\, (\chi_{Z^*_{1\nu i}}(\delta/2) + \chi_{Z^*_{2\nu i}}(\delta/2))\, |P_\nu]$$

$$\leq 4 \sum_{i=1}^{\nu} E([Z_{1\nu i}^{*2} \chi_{Z_{1\nu i}^*}(\delta/2) + Z_{2\nu i}^{*2} \chi_{Z_{2\nu i}^*}(\delta/2)] \mid P_\nu).$$

So, it is sufficient to show that $E[\sum_{i=1}^{\nu} Z_{j\nu i}^{*2}\chi_{Z_{j\nu i}^*}(\delta/2)|P_\nu] \to 0$ as $\nu \to \infty$ for j=1,2 (these conditions being equivalent to the Lindeberg condition of the classical Central Limit Theorem for $\sum_{i=1}^{\nu} Z_{1\nu i}^*$ and $\sum_{i=1}^{\nu} Z_{2\nu i}^*$). The proof is exactly the same as in Lemma 4.4 and hence is omitted.

Using now LeCam's third lemma (see [12], p. 208), lemma 4.6 and the remarks just above it, we get, under Q_ν, $\underset{\sim}{e}'\underset{\sim}{T}_\nu^*$ is asymptotically normal $((\underset{\sim}{e}'\underset{\sim}{\Sigma}_{00}\underset{\sim}{\gamma})\int_0^1 \phi(u)\,\phi^*(u)du,\ (\underset{\sim}{e}'\,\underset{\sim}{\Sigma}_0\,\underset{\sim}{e})\,I(g))$ for every $\underset{\sim}{e} \neq \underset{\sim}{0}$ of finite real constants i.e. $\underset{\sim}{T}_\nu^*$ (and hence $\underset{\sim}{S}_\nu$) is under Q_ν asymptotically normal $(\underset{\sim}{\Sigma}_{00}\,\underset{\sim}{\gamma}\int_0^1 \phi(u)\,\phi^*(u)\,du,\ \underset{\sim}{\Sigma}_0\,I(g))$. Also, from lemma (4.5), $\underset{\sim}{V}_\nu \to \underset{\sim}{\Sigma}_0\,I(g)$ in probability. Recalling definition of M_ν in (2.8) and using Slutsky's theorem, we get that under Q_ν, M_ν is distributed asymptotically as a non-central chi-square with p degrees of freedom and non-centrality parameter
$(\underset{\sim}{\gamma}'\,\underset{\sim}{\Sigma}_{00}'\,\underset{\sim}{\Sigma}_0^{-1}\,\underset{\sim}{\Sigma}_{00}\,\underset{\sim}{\gamma})\,(\int_0^1 \phi(u)\,\phi^*(u)du)^2/\,I(g)$.

5. Asymptotic optimality of tests and ARE . Let

(5.1) $\lambda_\nu = [L(\underset{\sim}{Z}_\nu|\underset{\sim}{\gamma})_{\underset{\sim}{\gamma}=\underset{\sim}{0}}] \,/\, [L(\underset{\sim}{Z}_\nu|\underset{\sim}{\gamma})_{\underset{\sim}{\gamma}=\hat{\underset{\sim}{\gamma}}_\nu}]$,

($\hat{\underset{\sim}{\gamma}}_\nu = (\hat{\gamma}_{1\nu},\ldots,\hat{\gamma}_{p\nu})'$ being the maximum likelihood estimator (m.l.e.) of $\underset{\sim}{\gamma}$) be the likelihood ratio test criterion for the testing problem considered here. Let,

(5.2) $M_\nu^* = (\underset{\sim}{T}_\nu'\,\underset{\sim}{\Sigma}^{-1}\,\underset{\sim}{T}_\nu)/\,I(f_{20})$.

If we make the assumptions on the likelihood function $L(\underset{\sim}{Z}_\nu|\underset{\sim}{\gamma})$ as in Wald [17], (see also [9]), which are not restated here for the sake of brevity, we get the following result:

THEOREM 5.1. $M_\nu^* + 2\log_e \lambda_\nu \to 0$ in P_ν-probability.

PROOF. Same as in Theorem 2.4. of [9] with ν, $\underset{\sim}{Z}_\nu$, $\underset{\sim}{\Sigma}$, $I(f_{20})$ and M_ν^* replacing N_ν, $\underset{\sim}{X}_\nu^*$, $\underset{\sim}{\Lambda}_\nu$, $A^2(F, \{I_j\})$ and S_ν respectively.

The theorem implies that $M_\nu^* + 2 \log_e \lambda_\nu \to 0$ in Q_ν-probability (because of contiguity). So if the assumptions (1.3), (1.4), (1.5), (1.8) along with the regularity assumptions of Wald are satisfied, a test procedure similar to the one we have considered based on M_ν^* possesses the optimal properties of the likelihood ratio test as described in Theorem 2.3. of [9].

We might also remark that using LeCam's third lemma, it can be shown that under the alternatives, M_ν^* is asymptotically a non-central chi-square with p degrees of freedom and non-centrality parameter, $(\underset{\sim}{\gamma}' \underset{\sim}{\Sigma} \underset{\sim}{\gamma})\, I(f_{20})$. Hence, the ARE (in the sense of Hannan [13]) of our proposed test based on M_ν relative to the one based on M_ν^* will be given by

$$e = [(\underset{\sim}{\gamma}' \underset{\sim}{\Sigma}'_{00} \underset{\sim}{\Sigma}_0^{-1} \underset{\sim}{\Sigma}_{00} \underset{\sim}{\gamma})/(\underset{\sim}{\gamma}' \underset{\sim}{\Sigma} \underset{\sim}{\gamma})]\, \rho_2^2 \quad , \tag{5.3}$$

where $\rho_2 = \int_0^1 \phi(u)\, \phi^*(u)du/[(\int_0^1 \phi^2(u)du)^{1/2}(\int_0^1 \phi^{*2}(u)du)^{1/2}]$.

We may also observe that in the particular case, $\phi^* = \phi$, $\psi_k^* = \psi_k$, for all $k=1,\ldots,p$, $e = 1$ i.e. the test procedure based on the statistic M_ν as we have proposed is asymptotically optimal in Wald's sense.

We may note that the general efficiency expression depends on $\underset{\sim}{\gamma} = \sigma^{-1} \underset{\sim}{\tau}$. However, using a well-known theorem of Courant (see Bodewig [1956]), we get,

$$\max_{\underset{\sim}{\gamma}} e = \text{maximal eigenvalue of } \underset{\sim}{\Sigma}^{-1} \underset{\sim}{\Sigma}'_{00} \underset{\sim}{\Sigma}_0^{-1} \underset{\sim}{\Sigma}_{00} \; ; \tag{5.4}$$

$$\min_{\underset{\sim}{\gamma}} e = \text{minimal eigenvalue of } \underset{\sim}{\Sigma}^{-1} \underset{\sim}{\Sigma}'_{00} \underset{\sim}{\Sigma}_0^{-1} \underset{\sim}{\Sigma}_{00} \; .$$

It is interesting to observe that instead of considering the rank statistics $S_{k\nu}(k=1,\ldots,p)$ as defined in (2.2) if we consider the mixed statistics

$S_{k\nu}^0 = \nu^{-1/2} \sum_{i=1}^{\nu} X_{\nu i}^{(k)}\, [a_\nu(R_{\nu i}^{(0)}) - \bar{a}_\nu]$, $k=1,\ldots,p$, (which would be more logical to consider when the $X_{\nu i}^{(k)}$ $(i=1,\ldots,\nu,\; k=1,\ldots,p)$ are observable,) the appropriate statistic will be the corresponding quadratic form in $S_{k\nu}^0$'s and the efficiency expression will be

$$e' = \rho_2^2 \; . \tag{5.5}$$

Thus in the particular case when $\phi^* = \phi$, the resulting test is optimal according to criteria described earlier. However, as already mentioned, the mixed statistics cannot be used when the $X_{\nu i}^{(k)}$ are not observable. We may also remark that unlike M_ν, the statistic based on $X_{\nu i}^{(k)}$'s is not unconditionally distribution-free when p=1.

We next investigate some particular cases where the efficiency expression can be simplified considerably leading to some interesting results.

Case I $p = 1$. Here, $e = \rho_1^2 \rho_2^2$, where,

$$\rho_1 = \int_0^1 \psi_1(u)\, \psi_1^*(u)du/[(\int_0^1 \psi_1^2(u)du)^{1/2}(\int_0^1 \psi_1^{*2}(u)du)^{1/2}] \quad (5.6)$$

We may note that this efficiency expression is the same as we would have obtained by an extension of Hájek's ideas to simple linear regression with stochastic predictors. The test statistic which we would consider in that case will be simply $S_{1\nu}$. This will be unconditionally distribution-free since in this situation under the hypothesis of no regression (or independence according to our model), the permutation distribution of $S_{1\nu}$ becomes identical with its unconditional distribution. For two-sided alternatives, the test procedure is optimal in Wald's sense. For one-sided alternatives, this will be asymptotically uniformly most powerful (UMP) as in Hájek [1962] in case $\phi^* = \phi$ and $\psi_1^* = \psi_1$. It might also be remarked that the test considered here is an alternative approach to the study of independence of two random variables as has been considered by Bhuchongkul [2]. Bhuchongkul followed the Chernoff-Savage [7] approach in proving the asymptotic normality of her test statistic and as such had to put rather stringent conditions regarding the existence of the derivatives of the score function and on its bounds. But unlike our case the asymptotic normality of her statistic follows even when regression is non-linear. However, her observation that the normal scores test is the locally most powerful rank test under the alternative that the variables under consideration have a bivariate normal distribution is equally valid in our case. In fact, in case of linear regression our result is slightly more general in the sense that whatever be the parent distribution, if $\psi_1^* = \psi_1$ and $\phi^* = \phi$, the resulting test as considered by us is asymptotically optimal. We give expressions for e in some particular cases:

TABLE 5.1. Showing some efficiency expressions.

$$\phi(u) = \sigma_0^{-1}\, \Phi^{-1}(u), \quad \psi_1(u) = \sigma_1\, \Phi^{-1}(u), \quad \Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp.(-t^2/2)dt$$

ϕ^*	$2u - 1$	$2u - 1$	$\sigma_{00}^{-1}\, \Phi^{-1}(u)$	$\sigma_{00}^{-1}\, \Phi^{-1}(u)$
ψ_1^*	$u - \frac{1}{2}$	$\sigma_2\, \Phi^{-1}(u)$	$u - \frac{1}{2}$	$\sigma_2\, \Phi^{-1}(u)$
e	$(3/\pi)^2$	$3/\pi$	$3/\pi$	1

Case II. p=2

(i) $\psi_k^*(u) = u - \frac{1}{2}$ (k=1,2), $\psi_k(u) = \sigma_k\, \Phi^{-1}(u)$, (k=1,2)

$\phi^*(u) = 2u-1$, $\phi(u) = \sigma_0^{-1}\, \Phi^{-1}(u)$, $\rho_{12} = \text{corr}(X_{\nu 1}^{(1)}, X_{\nu 1}^{(2)})$.

Here $\rho_2^2 = 3/\pi$ and we get after some simplifications ,

$$\underset{\sim}{\Sigma}_{00} = (2\sqrt{\pi}\,)^{-1} \begin{bmatrix} \sigma_1 & \sigma_2\rho_{12} \\ \sigma_1\rho_{12} & \sigma_2 \end{bmatrix} ;$$

$$\underset{\sim}{\Sigma}_0 = \begin{bmatrix} 1/12 & (1/2\pi)\sin^{-1}(\rho_{12}/2) \\ (1/2\pi)\sin^{-1}(\rho_{12}/2) & 1/12 \end{bmatrix}$$

$$\underset{\sim}{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

using these, we get after some lengthy algebra,

$e = (3/\pi)\ (\underset{\sim}{\delta}'\ \underset{\sim}{\Gamma}\ \underset{\sim}{\delta}) / (\underset{\sim}{\delta}'\ \underset{\sim}{\Gamma}_0\ \underset{\sim}{\delta})$,

where, $\underset{\sim}{\delta}' = (\gamma_1\sigma_1, \gamma_2\sigma_2)$, $\underset{\sim}{\Gamma}_0 = \begin{bmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{bmatrix}$ and $\underset{\sim}{\Gamma} = \underset{\sim}{\Gamma}_0 \begin{bmatrix} \pi/3 & 2\sin^{-1}(\rho_{12}/2) \\ 2\sin^{-1}(\rho_{12}/2) & \pi/3 \end{bmatrix}^{-1} \underset{\sim}{\Gamma}_0$

Then,

$$e_M = \max_{\underset{\sim}{\delta}} e = (3/\pi)^2\ (1+\rho_{12})/(1+(6/\pi)\sin^{-1}(\rho_{12}/2)),\ 0 \le \rho_{12} < 1$$

$$= (3/\pi)^2\ (1-\rho_{12})/(1-(6/\pi)\sin^{-1}(\rho_{12}/2)),\ -1 < \rho_{12} \le 0$$

$$e_m = \min_{\underset{\sim}{\delta}} e = (3/\pi)^2\ (1-\rho_{12})/(1-(6/\pi)\sin^{-1}(\rho_{12}/2)),\ 0 \le \rho_{12} < 1$$

$$= (3/\pi)^2\ (1+\rho_{12})/(1-(6/\pi)\sin^{-1}(\rho_{12}/2)),\ -1 < \rho_{12} \le 0$$

We can draw the following conclusions as in Bickel [3].

(1) $.912 \le e_M \le .917$. As a function of ρ, e_M is symmetric about $\rho = 0$, concave for $0 \le \rho < 1$ and $-1 < \rho \le 0$, attains its minimum of $(3/\pi)^2$ at $\rho = 0$, approaches it as $|\rho| \to 1$ and attains its maximum of .917 at $\rho = .78$.

(2) e_m considered as a function of ρ is symmetric about 0 and decreases from a maximum of $(3/\pi)^2$ at $\rho = 0$ to an infimum of $(3/\pi)\sin(\pi/3) = .827$ as $\rho \to 1$.

(ii) $\psi_k^*(u) = u - \frac{1}{2}$, (k=1,2), $\psi_k(u) = \sigma_k\ \Phi^{-1}(u)$, (k=1,2)

$\phi^*(u) = \sigma_{00}^{-1}\ \Phi^{-1}(u)$, $\phi(u) = \sigma_0^{-1}\ \Phi^{-1}(u)$.

Here $\rho_2^2 = 1$ and so $e = \rho_1^2$. Hence for e_M, e_m we get exactly the same expressions as in Bickel (1965) and hence, all his conclusions hold in this case.

(iii) $\psi_k^*(u) = \sigma_k^*\ \Phi^{-1}(u)$, $\psi_k(u) = \sigma_k\ \Phi^{-1}(u)$,

$\phi^*(u) = 2u-1$, $\phi(u) = \sigma_0^{-1}\ \Phi^{-1}(u)$.

Here, $\rho_2^2 = 3/\pi$ and $\rho_1^2 = 1$. Hence,

$$e = 3/\pi = .955$$

(iv) $\psi_k^*(u) = \sigma_k^*\ \Phi^{-1}(u)$, $\psi_k(u) = \sigma_k\ \Phi^{-1}(u)$, (k=1,2),

$\phi^*(u) = \sigma_{00}^{-1}\ \Phi^{-1}(u)$, $\phi(u) = \sigma_0^{-1}\ \Phi^{-1}(u)$. Then,

$$e = 1.$$

It can be shown in the general situation that if $\psi_k^*(u) = \sigma_k^*\ \Phi^{-1}(u)$, $\psi_k(u) = \sigma_k\ \Phi^{-1}(u)$, (k=1,...,p), then $\rho_1^2 = 1$ so that $e = \rho_2^2$. Also, when, $\psi_k^*(u) = u - \frac{1}{2}$, $\psi_k(u) = \sigma_k\ \Phi^{-1}(u)$, (k=1,...,p), it can be found as in the case p=2,

$$e = \rho_2^2\ (\underset{\sim}{\delta}'\ \underset{\sim}{\Gamma}\ \underset{\sim}{\delta})/\ (\underset{\sim}{\delta}'\ \underset{\sim}{\Gamma}_0\ \underset{\sim}{\delta}), \tag{5.7}$$

where $\underset{\sim}{\delta} = (\delta_1,\ldots,\delta_p)' = (\gamma_1\sigma_1,\ldots, \gamma_k\sigma_k)'$, $\underset{\sim}{\Gamma} = \underset{\sim}{\Gamma}_0 \underset{\sim}{\nabla}^{-1} \underset{\sim}{\Gamma}_0$,

where, $\underset{\sim}{\Gamma}_0 = ((\rho_{kk'}))$, $\underset{\sim}{\nabla} = (((2 \sin^{-1}(\frac{1}{2} \rho_{kk'})))$. We shall next show that

$\underset{\sim}{\Gamma}_0^{-1} - \underset{\sim}{\nabla}^{-1}$ is positive definite.

To prove this, it is sufficient to show that

$\underset{\sim}{\tau}' \underset{\sim}{\Gamma}_0 \underset{\sim}{\tau} \leq \underset{\sim}{\tau}' \underset{\sim}{\nabla} \underset{\sim}{\tau}$ for all $\underset{\sim}{\tau} \neq \underset{\sim}{0}$ (see e.g. Fraser [8], p. 55).

But $\underset{\sim}{\tau}' \underset{\sim}{\nabla} \underset{\sim}{\tau} - \underset{\sim}{\tau}' \underset{\sim}{\Gamma}_0 \underset{\sim}{\tau} = \sum_{j,j'=1}^{p} \tau_j \tau_{j'} [2 \sin^{-1}(\rho_{jj'}/2) - \rho_{jj'}]$

$$= \sum_{j,j'=1}^{p} \tau_j \tau_{j'} \sum_{n=2}^{\infty} c_n \rho_{jj'}^{2n-1}$$, where c_n are positive constants and is the same for all $j,j'=1,\ldots,p$. Thus

$$\underset{\sim}{\tau}' \underset{\sim}{\nabla} \underset{\sim}{\tau} - \underset{\sim}{\tau}' \underset{\sim}{\Gamma}_0 \underset{\sim}{\tau} = \sum_{n=2}^{\infty} c_n \sum_{j,j'=1}^{p} \tau_j \tau_{j'} \rho_{jj'}^{2n-1} . \tag{5.8}$$

Using now a result of I. Schur (see [1], p. 96) we get that if $A = ((a_{jj'}))$ and $B = ((b_{jj'}))$ are positive definite matrices, $C = ((c_{jj'})) = ((a_{jj'}b_{jj'}))$ is positive definite. Hence, since $((\rho_{jj'}))$ is positive definite, $((\rho^r_{jj'}))$ is positive definite for all $r=1,2,\ldots$. Thus from (5.7),

$$\underset{\sim}{\tau}' \underset{\sim}{\nabla} \underset{\sim}{\tau} - \underset{\sim}{\tau}' \underset{\sim}{\Gamma}_0 \underset{\sim}{\tau} > 0.$$

Since $\underset{\sim}{\Gamma}_0^{-1} - \underset{\sim}{\nabla}^{-1}$ is positive definite,

$$\underset{\sim}{\delta}' \underset{\sim}{\Gamma}_0(\underset{\sim}{\Gamma}_0^{-1} - \underset{\sim}{\nabla}^{-1}) \underset{\sim}{\Gamma}_0 \underset{\sim}{\delta} > 0, \text{ for all } \underset{\sim}{\delta} \neq \underset{\sim}{0}$$

i.e. $\underset{\sim}{\delta}' \underset{\sim}{\Gamma}_0 \underset{\sim}{\delta} > \underset{\sim}{\delta}' \underset{\sim}{\Gamma} \underset{\sim}{\delta}$. Hence, from (5.7), $e < \rho_2^2$.

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REFERENCES

[1] Bellman, R. (1960). Introduction to Matrix Analysis. McGraw Hill, New York.

[2] Bhuchongkul, S. (1964). A class of nonparametric tests for independence in bivariate populations. Ann. Math. Statist. 35 138-149.

[3] Bickel, P.J. (1965). On some asymptotically nonparametric competitors of Hotelling's T^2. Ann. Math. Statist. 36 160-173.

[4] Bodewig, E. (1956). Matrix Calculus. Interscience.

[5] Brillinger, D. R. (1962). A note on the rate of convergence of a mean. Biometrika. 49 574-576.

[6] Chatterjee, S.K. and Sen, P.K. (1964). Nonparametric tests for the bivariate two sample location problem. Calcutta Statist. Asscn. Bull. 13 18-58.

[7] Chernoff, H. and Savage, I.R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. Ann. Math. Statist. 29 972-994.

[8] Fraser, D.A.S. (1957). Nonparametric Methods in Statistics. Wiley, New York.

[9] Ghosh, M. (1969). An asymptotically optimal nonparametric test for grouped data. Institute of Statistics Mimeo Series No. 605, University of North Carolina, Chapel Hill, North Carolina.

[10] Hájek, J. (1961). Some extensions of the Wald-Wolfowitz-Noether Theorem. Ann. Math. Statist. 32 506-523.

[11] Hájek, J. (1962). Asymptotically most powerful rank order tests. Ann.Math. Statist. 33 1124-1147.

[12] Hájek, J. and Šidák, Z. (1967). Theory of Rank Tests. Academic Press, New York.

[13] Hannan, E.J. (1956). The asymptotic power of tests based on multiple correlation. J. Roy. Statist. Soc. Ser. B. 18 227-233.

[14] Hoeffding, W. (1951). A combinatorial central limit theorem. Ann. Math. Statist. 22 558-566.

[15] Loève, M. (1963). Probability Theory (3rd Edition). Van Nostrand, Princeton.

[16] Puri, M.L. and Sen, P.K. (1966). On a class of multivariate multisample rank order tests. Sankhyā. 28 353-376.

[17] Wald, A. (1943). Tests of Statistical Hypothesis concerning several parameters when the number of observations is large. Trans. Amer. Math. Soc. 54 426-482.