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A TABLE OF PERCENTAGE POINTS OF THE SMALLEST LATENT ROOT OF A 2 x 2 WISHART MATRIX

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The letter t_2 will denote a random variable having the distribution of the smallest latent root of a 2 x 2 standard Wishart matrix of n degrees of freedom and $t_2(\beta)$ will denote the number exceeded by t_2 with probability β. The numbers $t_2(\beta)$ are required in the construction of tolerance regions for the bivariate normal distribution (John, 1968 and Kleinbaum and John, 1969). The table of $t_2(\beta)$ given below might prove useful in other applications too, e.g., in a test of a hypothesis about the covariance matrix. (A table of percentage points of the distribution of the largest latent root of a Wishart matrix has been prepared by Sugiyama, 1968).

The number $t_2(\beta)$ is the value of t satisfying the equation

$$1-F_{2n}(2t) - \left[\Gamma(\tfrac{1}{2})/\Gamma(\tfrac{n}{2})\right] (\tfrac{t}{2})^{(n-1)/2} e^{-t/2}\left[1-F_{n+1}(t)\right] = \beta , \qquad (1)$$

where $F_n(t)$ is the chi square distribution function of n degrees of freedom, since the expression on the left-hand side is the probability with which t_2 exceeds t (John, 1963). (The differences between the present notations and the notations of John (1963) should be noted.) In order to solve equation (1), put it in the form

$$t = 2\left[\frac{\Gamma(n/2)}{\Gamma(1/2)} \frac{1-\beta-F_{2n}(2t)}{1-F_{n-1}(t)} e^{t/2}\right]^{2/(n-1)} . \qquad (2)$$

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The right-hand side of (2) evaluated at an approximate value of $t_2(\beta)$ yields an improved approximation to $t_2(\beta)$. A value of $t_2(\beta)$ of the desired accuracy can be obtained by repeating the process until two successive values agree to the required number of decimal places.

In the table, for $n > 30$, the value of $t_2(\beta)$ has been given only for values of n that are multiples of 10. Fairly accurate values of $t_2(\beta)$ for other values of n in the range [30, 100] can be obtained by interpolation. Somewhat greater accuracy can be achieved by interpolation for $\{t_2(\beta)/\chi^2_{n-1}(\sqrt{\beta})\}$, where $\chi^2_{n-1}(\sqrt{\beta})$ is the number exceeded with probability $\sqrt{\beta}$ by a chi square of n-1 degrees of freedom. This procedure was suggested by a result derived by one of the authors (S. J.), viz., that for large n, t_2 is distributed approximately as the smaller of two independent chi squares of (n-1) degrees of freedom. Multiplying by $\chi^2_{n-1}(\sqrt{\beta})$ the value of $\{t_2(\beta)/\chi^2_{n-1}(\sqrt{\beta})\}$ obtained by interpolation, one gets the value of $t_2(\beta)$. This approximation has been found to be superior to approximating the distribution of t_2 by $c\chi^2_f$, a multiple of a chi square of f degrees of freedom, where c and f are so chosen that $c\chi^2_f$ and t_2 agree in their first two moments. If $n > 100$, interpolation may be done by regarding $\{t_2(\beta) / \chi^2_{n-1}(\sqrt{\beta})\}$ as a function of (1/n) and taking the value of $t_2(\beta) / \chi^2_{n-1}(\sqrt{\beta})$ for $n = \infty$, i.e. $1/n = 0$, to be unity.

TABLE: Lower Percentage Points of the Smallest Latent Root of a 2 x 2 Wishart Matrix[2]

$n \backslash \beta$	.99	.95	.90
2	.00006377	.001594	.06406
3	.01005	.05129	.10536
4	.06477	.19795	.32802
5	.18123	.43132	.64290
6	.35731	.73320	1.02533
7	.58582	1.08926	1.45927
8	.85945	1.48920	1.93393
9	1.17197	1.92535	2.44174
10	1.51815	2.39204	2.97719
11	1.89378	2.88483	3.53611
12	2.29536	3.40037	4.11536
13	2.72005	3.93580	4.71237
14	3.16540	4.48898	5.32502
15	3.62943	5.05792	5.95167
16	4.11036	5.64119	6.59089
17	4.60681	6.23734	7.24152
18	5.11736	6.84538	7.90254
19	5.64099	7.46420	8.57304
20	6.17672	8.09210	9.25232

(con't.)

TABLE (con't.)

$n \backslash \beta$	.99	.95	.90
21	6.72368	8.73115	9.93969
22	7.28107	9.37791	10.63455
23	7.84822	10.03255	11.33634
24	8.42453	10.69481	12.04476
25	9.00939	11.36400	12.75937
26	9.60242	12.03983	13.47963
27	10.20303	12.72190	14.20535
28	10.81089	13.40983	14.93617
29	11.42566	14.10335	15.67184
30	12.04691	14.80228	16.41201
40	18.55818	22.03159	24.02420
50	25.48186	29.59274	31.92359
60	32.69940	37.38909	40.02634
70	40.13982	45.36345	48.28289
80	47.75737	53.47865	56.66093
90	55.51891	61.70876	65.13785
100	63.40167	70.03507	73.69821

[2] The values in the table give $t_2(\beta)$ where

$$\Pr\{t_2 > t_2(\beta)\} = \beta.$$

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