Section 1

Statistics of Market Rates
The bulk of OTC derivatives consist of \textit{interest rate swaps} and \textit{cross-currency swaps}. A cross currency swap is similar to a (single currency) interest rate swap, except that the payments made by the two parties are in different currencies. Either side may involve fixed interest payments or interest payments tied to a floating rate. There is an agreed “notional principal” amount in each currency, and typically the swap starts and finishes with the actual exchange of these sums.
The notional principals are usually chosen to be equal in value at the inception of the swap, but changes in foreign exchange (FX) rates may make them very unequal at its maturity. The “mark to market” value of the swap therefore depends on FX rates. If either side of the swap carries a fixed rate, the value also depends on the yield curve in the corresponding currency. A floating rate side is relatively insensitive to yield curve changes.
One simple kind of cross-currency swap is the *FX forward contract*. This is an agreement to exchange specified sums in two currencies on a specified future date. It therefore involves only a single cash flow on each side, at maturity. Because its value depends on the FX rate and the zero-coupon yields for the specified maturity in both currencies, the value for a long-term forward is very volatile.
To avoid the yield-curve sensitivity, long term hedging of FX risk is usually effected through a cross-currency swap with floating rate payments on both sides—a *basis swap*. Although the cash flows are more complicated than those for a forward, the elimination of yield-curve sensitivity usually more than compensates for the additional complication.
These two basic types of swap may account for 90% of the volume of a swaps market participant. The balance would typically be options of one sort or another: FX options, that is puts and calls on FX rates; interest rate caps and floors; and swaptions, that is options to enter swaps. The values of options depend on volatilities of underlying rates as well as on the rates themselves.
Foreign Exchange Rates

Figure 1 shows the JPY/USD exchange rate series from January, 1995, to the start of 1998.

Figure: Dollar/yen exchange rate.
The rate is expressed as the price of $1.00 in Yen, and measures the strength of the US dollar relative to the yen; the low around April, 1995, reflected a rapid slide in the dollar that started in March, followed by a matching recovery.
It seems reasonable to study the day-to-day changes in the rate, and the logarithm of the ratio of one day’s rate to the previous rate, shown in Figure 2, suggests itself.

![Graph showing logarithmic returns, dollar/yen exchange rate.](image-url)
Figure 3 shows a *qq*-plot of these differences against the normal distribution (that is, a graph of the quantiles of the empirical distribution of logarithmic differences *versus* matching quantiles of the normal or Gaussian distribution).

![Normal Q–Q Plot](image)

**Figure**: Quantile-quantile plot of logarithmic differences of dollar/yen exchange rate.
The distribution is substantially longer tailed than the Gaussian, as indicated by the distinct curvature.
The next question concerns the correlation structure of the series. Figure 4 shows the correlogram of the series of logarithmic differences, with lines indicating the limits for testing at the 5\% level the hypothesis that the autocorrelation is zero.

**Figure**: Correlogram of logarithmic differences of dollar/yen exchange rate.
Only one correlation (out of 28) appears to be definitely significant at the 5% level, in line with what we would expect under the null hypothesis. Interestingly, the first correlation, at lag 1, is marginally significant and negative. This suggests that the market overreacts to an event on a given day, and retreats slightly on the following day. However, one would not want to over-interpret the graph.
The original graph (Figure 1) suggests that in the long term the series may be subject to trends that are sustained for some months. If this were true it would lead to small positive autocorrelations for several lags, and there is in fact a preponderance of positive (though insignificant) correlations in the graph.
A good way to explore such effects is through the *power spectrum*. Figure 5 shows an estimate of the power spectrum of the series of logarithmic differences.

**Figure**: Power spectrum of logarithmic differences of dollar/yen exchange rate.
The spectrum units are *decibels*, or $10 \times \log_{10}(\text{spectrum})$; 1 decibel corresponds to a factor of $1.26$ in the power spectrum. The vertical line towards the right hand side of the graph is a (pointwise) 95% confidence interval for the spectral density, and indicates that the data are consistent with a flat spectrum, or in other words that the logarithmic differences appear to be white noise (having no serial correlation).
Sustained trends on a time-scale of months or years would have been expected to introduce a peak in the spectrum at low frequencies. In the graph, frequencies are measured on a scale of cycles per day, so a monthly cycle would occur at the frequency 1 cycle per month or roughly 0.05 cycles per day.
A trend sustained for one month would not correspond to any exact cycle, but would be broken down across frequencies from zero to roughly one half this, or 0.025 cycles per day. The absence of any such peak (at least, one that reaches higher than peaks at much higher frequencies) means that the trends in the original figure are no stronger than one would expect on the basis of a random walk: accumulated white noise.
Conditional Heteroscedasticity

We have seen that the JPY/USD exchange rate exhibits nearly uncorrelated day-to-day logarithmic changes. However, this does not mean that the change from one day to the next is independent of the past. In fact, the magnitudes of the changes show a certain degree of dependence.
Figure 6 shows the correlogram of the absolute values of the changes.

Figure: Correlogram of absolute values of the logarithmic differences of dollar/yen exchange rate.
Figure 7 shows an estimate of the corresponding power spectrum.

![Power Spectrum](image)

**Series**: abs(jyyd)

**Smoothed Periodogram**

*bandwidth = 0.0139*

**Figure**: Power spectrum of absolute values of the logarithmic differences of dollar/yen exchange rate.
These graphs show that there is some degree of persistence in the volatility of the day-to-day changes. For instance, the correlation between a given absolute change and the average of the preceding 10 days’ absolute changes is around 0.2.
Furthermore, if we examine the distribution of the ratio of a given change to the average of the preceding 10 days’ absolute changes, the \( qq \)-plot is as shown in Figure 8.

![Normal Q–Q Plot](image)

**Figure**: Quantile-quantile plot of ratio of logarithmic differences of dollar/yen exchange rate to the average of the preceding 10 absolute differences.
This is somewhat straighter than the plot for the original data, except perhaps for a small number of points at each end, and suggests that it may be more reasonable to view the changes as being *conditionally* normally distributed, given the past values, than as *unconditionally* normally distributed. (Note that the partial straightening of the qq-plot cannot be an artefact: dividing a variable by another unrelated variable can only *increase* the heaviness of the tails.)
Figure 9 shows a qq-plot of the same ratios against the $t$-distribution with 5 degrees of freedom, and suggests that this distribution might provide a better model.

Figure: Quantile-quantile plot of ratio of logarithmic differences of dollar/yen exchange rate to the average of the preceding 10 absolute differences against the $t$-distribution with 5 degrees of freedom.
This persistence of volatility is called *conditional heteroscedasticity*. We might write

\[ e_t \mid e_{t-1}, e_{t-2}, \ldots \sim N(0, \sigma_t^2) \]

where \( \sigma_t \) is some function of \( e_{t-1}, e_{t-2}, \ldots \), or more constructively

\[ e_t = \sigma_t z_t \]

where \( z_t \sim N(0, 1) \) and the \( z \)'s are independent of each other. Again, a longer tailed distribution than the normal would provide a closer model.
ARCH and GARCH Models

The Auto-Regressive Conditionally Heteroscedastic (ARCH) model is related to this structure. It consists of an autoregressive structure for some observed time series \( \{y_t\} \), in which the errors \( \{e_t\} \) have the above structure:

\[
y_t = ay_{t-1} + e_t,
\]

for instance. In the original ARCH model, the function \( \sigma_t^2 \) was just a linear compound of recent \( e^2 \)'s:

\[
\sigma_t^2 = \omega + \sum_{r=1}^{q} \alpha_r e_{t-r}^2
\]
However, it was quickly recognized that this could be interpreted as having $\sigma^2_t$ depend in a Moving Average (MA) fashion on the $e^2$’s. The Generalized ARCH (GARCH) model extends this dependence to something similar to ARMA:

$$\sigma^2_t = \omega + \sum_{r=1}^{p} \beta_r \sigma^2_{t-r} + \sum_{r=1}^{q} \alpha_r e^2_{t-r}.$$ 

Of course, the underlying $z$’s need not be normally distributed. The $qq$-plot shown above shows some remaining heavy-tailedness; this might be attributed to inadequate modelling of the conditional variance, or could indicate that there is intrinsically nonGaussian behavior (including a hint of skewness).
Fitting a GARCH model requires identifying good values for $p$ and $q$ and estimating the $\alpha$’s and $\beta$’s, in addition to the modelling of the autoregressive part (the equation relating $y$’s to $e$’s), which could itself be in the general ARMA form. Identification and estimation are nontrivial exercises; modules for carrying out these steps are available for R, Splus, and SAS.
Fitting the GARCH(1, 1) model using the R function `garch` from the `tseries` library gives the output shown in Figures 10 and 11, which shows that

\[ \hat{\omega} = 7.995 \times 10^{-7}, \hat{\alpha}_1 = 0.03932, \hat{\beta}_1 = 0.9442. \]
Call:
garch(x = jyyd)

Model:
GARCH(1,1)

Residuals:

<table>
<thead>
<tr>
<th></th>
<th>Min</th>
<th>1Q</th>
<th>Median</th>
<th>3Q</th>
<th>Max</th>
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</thead>
<tbody>
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<td>jyyd</td>
<td>-6.0105</td>
<td>-0.5487</td>
<td>0.0000</td>
<td>0.5904</td>
<td>4.1413</td>
</tr>
</tbody>
</table>

Coefficient(s):

| Estimate | Std. Error | t value | Pr(>|t|) |
|----------|------------|---------|----------|
| a0       | 7.995e-07  | 1.221e-07 | 6.549 | 5.8e-11 *** |
| a1       | 3.932e-02  | 3.760e-03 | 10.458 | < 2e-16 *** |
| b1       | 9.442e-01  | 5.388e-03 | 175.263 | < 2e-16 *** |

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Signif. codes:  0 ***  0.001 **  0.01 *  0.05 .  0.1  1

Figure : GARCH(1, 1) output.
Diagnostic Tests:
Jarque Bera Test

data: Residuals
X-squared = 1086.034, df = 2, p-value < 2.2e-16

Box-Ljung test

data: Squared.Residuals
X-squared = 1.3951, df = 1, p-value = 0.2375

Figure: GARCH(1, 1) output, continued.
A GARCH(1, 1) model is stationary iff $\alpha_1 + \beta_1 < 1$, so this fitted model is close to non-stationarity. The *unconditional* variance is

$$\omega/[1 - (\alpha_1 + \beta_1)],$$

which is estimated here as $4.866 \times 10^{-5}$, for a daily volatility of 0.006975 and an annualized volatility of 0.1103 (the daily volatility multiplied by $\sqrt{250}$). The Jarque Bera test, a test for departures from the normal distribution, is not unexpectedly highly significant. The Box-Ljung test gives no evidence that the GARCH(1, 1) formulation is inadequate.
The \( qq \)-plot of the GARCH residuals against the \( t \)-distribution with 10 degrees of freedom is shown in Figure 12.

\[ \text{Figure} : \text{Quantile-quantile plot of GARCH residuals against the} \ t \text{-distribution with 10 degrees of freedom.} \]

The asymmetry of the distribution is now clear. The upper tail is fitted well, but the lower tail is longer than that of the matching \( t \)-distribution.