Partition based priors to analyze data arising from failure models and censoring models

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Summary

Failure and repair models
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Censoring models
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Partition based (PB) priors
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Failure and repair models

Censoring models

Partition based (PB) priors introduced by Sethuraman and Hollander (2008) turn out to be the natural priors for analyzing both these kinds of data.
Failure and repair models - I

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where $A_n$ is a set that depends on $X_1, \ldots, X_n, Y_1, Y_2, \ldots, Y_n$ for $n = 1, 2, \ldots$. 
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Repair models can also be viewed as search models.
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where $A_1 = (Z_1, \infty), A_2 = (Z_2, \infty), \ldots$. 
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Partition Based Prior Distributions

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Let $\mathcal{B} = (B_1, \ldots, B_m)$ be a finite partition of $\mathcal{X}$. Let $P(\mathcal{B}) = (P(B_1), \ldots, P(B_m))$ be the partition probability vector, and let $(P_{B_1}, \ldots, P_{B_m})$ be the vector of restricted pm’s.
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Let $\mathcal{B} = (B_1, \ldots, B_m)$ be a finite partition of $\mathcal{X}$. Let $P(\mathcal{B}) = (P(B_1), \ldots, P(B_m))$ be the partition probability vector, and let $(P_{B_1}, \ldots, P_{B_m})$ be the vector of restricted pm’s.

There is a one-to-one correspondence between pm’s $P$ and the pair $(P(\mathcal{B}), (P_{B_1}, \ldots, P_{B_m}))$ since

$$P(B) = \sum_{i=1}^{m} P(B_i) P_{A_i}(B).$$

The pair $(P(\mathcal{B}), (P_{B_1}, \ldots, P_{B_m}))$ is just another way to understand a pm $P$. 
Partition based Priors

Suppose that $P(B), P_{B_1}, \ldots, P_{B_m}$ are independent, with $P(B)$ having a pdf proportional to $h(y)$, and $P_{B_i}$ having a distribution $G_i, i = 1, \ldots, m$. 
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Then we say that $P$ has a Partition based (PB) prior distribution

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where $\mathcal{G} = G_1 \times \cdots \times G_m$. 
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The $G_i$’s are probability measures on the space of probability measures and there are $m$ of them. Hopefully, we will use standard nonparametric priors for $G_i$, $i = 1, \ldots, m$, so that we also know the corresponding posterior distributions $G_i^X$. 

Let $\alpha$ be a finite non-zero measure on $\mathcal{X}$ and take $G_i$ to be $D_{\alpha|A_i}$, $i = 1, \ldots, m$. 
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If $B^*$ is a subpartition of $B$, then a PB Dirichlet prior on $B$ is also a PB Dirichlet prior on $B^*$. 
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If $B^*$ is a subpartition of $B$, then a PB Dirichlet prior on $B$ is also a PB Dirichlet prior on $B^*$.

A PB Dirichlet prior $D(B, h, \alpha)$ is the Dirichlet prior $D_{\alpha}$ if and only if

$$h(y) \propto y_1^{\alpha(A_1) - 1} \cdots y_m^{\alpha(A_m) - 1}.$$
Posterior distributions in standard nonparametric Bayesian problem

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That is the unknown pm $P$ has prior distribution $G$, and the data $X$ given $P$ has distribution $P$. $G^X$ is simply the notation for the posterior distribution of $P$ given $X$. 
Consider the problem where $P$ has the PB prior $H(B, h, G)$ and the data $X$ given $P$ has distribution $P_A$. Suppose that $A = \bigcup_{i=1}^{r} B_i$, and define $y_A = \sum_{i=1}^{r} y_i$. Let $R$ be the random index such that $X \in B_R$; note that $R \in \{1, \ldots, r\}$. 
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Theorem

Then the posterior distribution of $P$ given $X$ is

$$H(\beta, h^X, G^X)$$

where

$$h^X(y) \propto h(y) \cdot \frac{y_R}{y_A}$$

and

$$G^X = G_1 \times \cdots G_{R-1} \times G_R^X \times G_{R+1} \times \cdots G_m.$$
Suppose that $P$ has the PB Dirichlet distribution $\mathcal{D}(\mathcal{B}, h, \alpha)$.

Given $P$, let the data $X$ have distribution $P_A$ where, as before, $A = \bigcup_{i=1}^{r} B_i$. 

The condition $A = \bigcup_{i=1}^{r} B_i$ is not required when the prior is a PB Dirichlet prior; the finite partition $\mathcal{B}$ can be enlarged to a new partition $\mathcal{B}^*$ by including $A$; the posterior distribution will still be a PB Dirichlet distribution on the partition on $\mathcal{B}^*$.

Even more; The partition $\mathcal{B}$ can be enlarged with the random restriction sets based on all the data, and still a blind application of the method above will give the correct answer. See Sethuraman and Hollander (2008).
Suppose that $P$ has the PB Dirichlet distribution $\mathcal{D}(B, h, \alpha)$. Given $P$, let the data $X$ have distribution $P_A$ where, as before, $A = \bigcup_{i=1}^{r} B_i$. Then the posterior distribution of $P$ given $X$ is

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Posterior distributions of PB priors with repair data - 3

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As an illustration, we will write a Dirichlet prior as a partition based prior on a partition determined by the data and compute the posterior; we will see that we still obtain the correct answer.
Illustration a Dirichlet Prior as a PB prior

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Then rewrite $D_{\alpha} = H(\mathcal{B}, h, D_{\alpha})$ with $h \propto y_1^{\alpha(A) - 1} y_1^{\alpha(A^c) - 1}$ and a blind application of the main theorem will show that the posterior is $H(\mathcal{B}, h^X, D_{\alpha+\delta_X})$, with $h^X \propto y_1^{\alpha(A)} y_1^{\alpha(A^c) - 1}$, which luckily reduces to the correct answer, namely $D_{\alpha+\delta_X}$. 
Bayes and Frequentist Estimates of the Survival Function

Bayes and W–S Estimates from AC Data Set (small) of Whittaker –Samaniego

Failure Age (Hours)
Survival Probability

Bayes Estimate
WS Estimate
PB Priors and Censored Data - 1

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Let us consider just only one potential observation $X$ and also assume that it is censored by $A$. (The uncensored case is the standard nonparametric Bayesian problem.)
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PB Dirichlet Priors and Censored Data

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In particular, suppose that $P$ has a Dirichlet prior $D_\alpha$. Let $A_1, \ldots, A_n$ be the censoring sets for the potential data $X_1, \ldots, X_n$, which are i.i.d. $P$. Let $B = (B_1, \ldots, B_m)$ be the partition based on $A_1, \ldots, A_n$. Let $Y_{A_i} = \sum_{j: B_j \subset A_i} y_j$, $i = 1, \ldots, n$. 

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**Theorem**

*Then the posterior distribution of $P$ given that all the data $X_1, \ldots, X_n$ are censored is*

$$D(B, h(y) \cdot y_{A_1} \cdots y_{A_n}, \alpha)$$

*where $h(y) \propto y_1^{\alpha(B_1) - 1} \cdots y_m^{\alpha(B_m) - 1}$.*
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where $h(y) \propto y_1^{\alpha(B_1)-1} \cdots y_m^{\alpha(B_m)-1}$, and this is a PB Dirichlet distribution.
The moral of this talk is that if you start even from a Dirichlet prior, and the data consists of general repair or general censored data, then the posterior will be a PB Dirichlet distribution.
Susarla-Van Ryzin Example -1

Susarla and Van Ruzin (1976) considered right censoring
Susarla and Van Ruzin (1976) considered right censoring (i.e. the censoring set $A_i = [a_i, \infty)$ among random variables on $[0, \infty)$). They also used a Dirichlet prior for $P$ and obtained the expectation of the df $F(x)$ under the posterior distribution.
Susarla and Van Ruzin (1976) considered right censoring (i.e. the censoring set $A_i = [a_i, \infty)$ among random variables on $[0, \infty)$).

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Their example has been revisited by many authors.
We will show how their example works with the use of PB priors.
Susarla-Van Ryzin Example -2

Their data set is

0.8, 1.0+, 2.7+, 3.1, 5.4, 7.0+, 9.2, 12.1+, 

where $a+$ denotes that the censoring set was $[a, \infty)$ and that potential observation was censored.
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The posterior distribution given the uncensored observations is Dirichlet with parameter $\alpha^* = \alpha + \delta_{0.8} + \delta_{3.1} + \delta_{5.4} + \delta_{9.2}$. We can take this to be the prior distribution and say that the remaining data $1.0+, 2.7+, 7.0+, 12.1+$,
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The partition formed by the censoring sets is

\[ \mathcal{B} = (B_1, \ldots, B_5) = ([0, a_1), [a_2, a_3), \ldots, [a_4, \infty)) \]

\[ = ([0, 1.0), [1.0, 2.7), [2, 7, 7.0), [7.0, 12.1), [12.1, \infty)). \]
Susarla-Van Ryzin Example - 3

The partition formed by the censoring sets is

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\[ = ([0, 1.0), [1.0, 2.7), [2, 7, 7.0), [7, 0, 12.1), [12.1, \infty)). \]

The posterior distribution given all the data is therefore

\[ \mathcal{D}(\mathcal{B}, h(y) Y_2 Y_3 Y_4 Y_5, \alpha^*) \]

where

\[ Y_j = y_j + \cdots + y_5, j = 2, \ldots, 5 \]

are the tail sums of the \( y \)'s and

\[ h(y) \propto \prod_{1}^{5} y_j^{\alpha^*(B_j)-1}. \]
The expectation of $P(B_j)$ under this posterior is the expectation of $y_j$ under the finite dimensional distribution with pdf proportional to $h(y) Y_2 Y_3 Y_4 Y_5$.

The transformation $y_j = z_j \prod_{r=1}^{j-1} (1 - z_r), j = 1, \ldots, 4$ makes $z_1, \ldots, z_4$ into independent Beta variables. This makes the calculation of the expected value of $P(B_j)$ very easy.
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The expected value of $P([a, \infty))$ can be computed just as easily for any $a \in B_i$ since

$$E(P([a, \infty))) = E(y_i)E(P_{B_i}([a, a_i])) + E(Y_{i+1})$$

(Explain independence)
Susarla-Van Ryzin Example - 4

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\[
E(P_{B_i}([a, a_i])) = \alpha^*([a, a_i]) / \alpha^*([a_{i-1}, a_i]).
\]
Suppose that we write the data from the example of Susarla-Van Ryzin as

\[ 0.8-,\ 1.0+,\ 2.7+,\ 3.1-,\ 5.4-,\ 7.0+,\ 9.2-,\ 12.1+. \]
Susarla-Van Ryzin Example, Extended -1

Two way censoring

Suppose that we write the data from the example of Susarla-Van Ryzin as

0.8−, 1.0+, 2.7+, 3.1−, 5.4−, 7.0+, 9.2−, 12.1+.

where $a−$ indicates that a potential data was censored to the interval $[0, a)$ (i.e. right censored).
Suppose that we write the data from the example of Susarla-Van Ryzin as

\[0.8-, 1.0+, 2.7+, 3.1-, 5.4-, 7.0+, 9.2-, 12.1+\]

where \(a-\) indicates that a potential data was censored to the interval \([0, a)\) (i.e. right censored).

The data generates a partition \(B = (B_1, \ldots, B_9)\) of 9 intervals.
If we use a Dirichlet prior $\mathcal{D}_\alpha$, it can also be viewed as PB Dirichlet on this partition. The prior distribution of $P(B) = (P(B_1), \ldots, P(B_9))$ is the finite dimensional Dirichlet with pdf proportional to

$$h(y) = \times_{i=1}^{9} y_i^{\alpha(B_i) - 1}.$$
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The posterior distribution of $P$ is a PB Dirichlet distribution

$$\mathcal{D}(B^*, h(y)Z_1Y_3Y_4Z_5Z_5Y_7Z_8Y_9, \alpha)$$

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where

$$Z_j = \sum_{i=1}^j y_i, \quad Y_j = \sum_{i=j}^9 y_i.$$ 

This also illustrates how to handle any kind of censoring in an arbitrary space.
A closed form expression for $E(y_j)$ under the posterior distribution is not available. However, it is just $E(h(y_j|z_1,y_3,y_4,z_5,z_5,y_7,z_8,y_9))$.

where $E_h$ denotes expectation under the finite dimensional Dirichlet distribution $D(\alpha(B_1),...,\alpha(B_9))$.

We can generate samples $(y_{r1},...,y_{r9})$, $r = 1,\ldots,N$ from $h(y_j)$ by using independent Gamma random variables and approximate the above as $\sum_{r=1}^{N} y_{rj}z_{r1}y_{r3}y_{r4}z_{r5}z_{r5}y_{r7}z_{r8}y_{r9} \sum_{r=1}^{N} z_{r1}y_{r3}y_{r4}z_{r5}z_{r5}y_{r7}z_{r8}y_{r9}$. 

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We can generate samples $(y_1^r, \ldots, y_9^r), r = 1, \ldots, N$ from $h(y)$ by using independent Gamma random variables and approximate the above as

$$\frac{\sum_{r=1}^N y_j^r Z_1^r Y_3^r Y_4^r Z_5^r Z_5^r Y_7^r Z_8^r Y_9^r}{\sum_{r=1}^N Z_1^r Y_3^r Y_4^r Z_5^r Z_5^r Y_7^r Z_8^r Y_9^r}.$$
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Of course, we can estimate many features other than just the distribution function, i.e, median, quantiles, etc.
Susarla-Van Ryzin Example, Extended -5

We use our method to recalculate the estimates of $F(t)$ based on Susarla-Van Ryzin data. When the transformation to independent Beta variables is used we get the same results as Susarla and Van Ryzin. However, this depends on the fact that all the censoring were right censoring results.

<table>
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We also use on LLN method to compute estimates of $F(t)$. A comparison of the results is given below:

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<tr>
<td>SSS</td>
<td>0.1083</td>
<td>0.1190</td>
<td>0.2084</td>
<td>0.3011</td>
<td>0.4706</td>
<td>0.5261</td>
<td>0.6802</td>
<td>0.7476</td>
</tr>
</tbody>
</table>
We use our method to recalculate the estimates of $F(t)$ based on Susarla-Van Ryzin data. When the transformation to independent Beta variables is used we get the same results as Susarla and Van Ryzin. However, this depends on the fact that all the censoring were right censoring results.

We also use on LLN method to compute estimates of $F(t)$. A comparison of the results is given below:

<table>
<thead>
<tr>
<th>t</th>
<th>0.80</th>
<th>1.00+</th>
<th>2.70+</th>
<th>3.10</th>
<th>5.40</th>
<th>7.00+</th>
<th>9.20</th>
<th>12.10+</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>0.1083</td>
<td>0.1190</td>
<td>0.2071</td>
<td>0.3006</td>
<td>0.4719</td>
<td>0.5256</td>
<td>0.6823</td>
<td>0.7501</td>
</tr>
<tr>
<td>LLN</td>
<td>0.1082</td>
<td>0.1189</td>
<td>0.2068</td>
<td>0.3000</td>
<td>0.4708</td>
<td>0.5244</td>
<td>0.6810</td>
<td>0.7487</td>
</tr>
<tr>
<td>SSS</td>
<td>0.1083</td>
<td>0.1190</td>
<td>0.2084</td>
<td>0.3011</td>
<td>0.4706</td>
<td>0.5261</td>
<td>0.6802</td>
<td>0.7476</td>
</tr>
<tr>
<td>KME</td>
<td>0.1250</td>
<td>0.1450</td>
<td>0.1250</td>
<td>0.3000</td>
<td>0.4750</td>
<td>0.4750</td>
<td>0.7375</td>
<td>0.7375</td>
</tr>
</tbody>
</table>
We take the data from Susarla-Van Ryzin and change the uncensored observations to be left censored. We use the same prior distribution for $P$ and obtain estimates of $F(t)$ using the LLN approximation.

\[ \hat{F}(t) \]

<table>
<thead>
<tr>
<th>$t$</th>
<th>LLN</th>
<th>0.80-</th>
<th>1.00+</th>
<th>2.70+</th>
<th>3.10-</th>
<th>5.40-</th>
<th>7.00+</th>
<th>9.20-</th>
<th>12.10+</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1737</td>
<td>0.1937</td>
<td>0.3625</td>
<td>0.3968</td>
<td>0.5274</td>
<td>0.5990</td>
<td>0.6796</td>
<td>0.7721</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Graphical Comparisons

Estimates of the df from censored (all types) data, based on PB priors

Two-way censoring

Prior df

Right censoring


